MATRIX ALGEBRA REVIEW

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PRELIMINARIES

A matrix is a way of organizing information.

It is a rectangular array of elements arranged in rows and columns. For example, the following matrix A has m rows and n columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

All elements can be identified by a typical element a_{ij}, where i=1,2,…,m denotes rows and j=1,2,…,n denotes columns.

A matrix is of order (or dimension) m by n (also denoted as (m x n)).
A matrix that has a single column is called a column vector.
A matrix that has a single row is called a row vector.

TRANSPOSE

The **transpose** of a matrix or vector is formed by interchanging the rows and the columns. A matrix of order (m x n) becomes of order (n x m) when transposed.

For example, if a (2 x 3) matrix is defined by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Then the transpose of A, denoted by A', is now (3 x 2)

$$A' = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$$

- $(A')' = A$
- $(kA)' = kA'$, where k is a scalar.

SYMMETRIC MATRIX

When $A' = A$, the matrix is called **symmetric**. That is, a symmetric matrix is a square matrix, in that it has the same number of rows as it has columns, and the off-diagonal elements are symmetric (i.e. $a_{ij} = a_{ji}$ *for all i and j*).

For example,

$$A = \begin{bmatrix} 4 & 5 & -3 \\ 5 & 7 & 2 \\ -3 & 2 & 10 \end{bmatrix}$$

A special case is the **identity matrix**, which has 1's on the diagonal positions and 0's on the off-diagonal positions.

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

The identity matrix is a **diagonal matrix**, which can be denoted by $diag(a_1, a_2, ..., a_n)$, where a_i is the ith element on the diagonal position and zeros occur elsewhere. So, we can write the identity matrix as $I = diag(1,1,...,1)$.

ADDITION AND SUBTRACTION

Matrices can be added and subtracted as long as they are of the same dimension. The addition of matrix A and matrix B is the addition of the corresponding elements of A and B. So, $C = A + B$ implies that $c_{ij} = a_{ij} + b_{ij}$ for all i and j.

For example, if

$$A = \begin{bmatrix} 2 & -3 \\ 6 & 10 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 6 \\ 5 & -8 \end{bmatrix}$$

Then

$$C = \begin{bmatrix} 2 & 3 \\ 11 & 2 \end{bmatrix}$$

- $A \pm B = B \pm A$
- $(A \pm B) \pm C = A \pm (B \pm C)$
- $(A \pm B)' = A' \pm B'$

MULTIPLICATION

If k is a scalar and A is a matrix, then the product of k times A is called scalar multiplication. The product is k times each element of A. That is, if $B = kA$, then $b_{ij} = ka_{ij}$ for all i and j.

In the case of multiplying two matrices, such as $C = AB$, where neither A nor B are scalars, it must be the case that

the number of columns of A = the number of rows of B

So, if A is of dimension (m x p) and B of dimension (p x n), then the product, C, will be of order (m x n) whose ijth element is defined as

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

In words, the ijth element of the product matrix is found by multiplying the elements of the ith row of A, the first matrix, by the corresponding elements of the jth column of B, the second matrix, and summing the resulting product. For this to hold, the number of columns in the first matrix must equal the number of rows in the second.

For example,

$$F = AD = \begin{bmatrix} 6 & 8 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 3 & -8 & 1 \\ 9 & 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 6*3+8*9 & 6*(-8)+8*2 & 6*1+8*5 \\ (-2)*3+4*9 & (-2)*(-8)+4*2 & (-2)*1+4*5 \end{bmatrix}$$

$$= \begin{bmatrix} 90 & -32 & 46 \\ 30 & 24 & 18 \end{bmatrix}$$

- A (m x 1) column vector multiplied by a (1 x n) row vector becomes an (m x n) matrix.
- A (1 x m) row vector multiplied by a (m x 1) column vector becomes a scalar.
- In general, $AB \neq BA$.
- But, $kA = Ak$ if k is a scalar and A is a matrix.
- And, $AI = IA$ if A is a matrix and I is the identity matrix and conformable for multiplication.

The product of a row vector and a column vector of the same dimension is called the **inner product** (also called the dot product), its value is the sum of products of the components of the vectors. For example, if j is a (T x 1) vector with elements 1, then the inner product, j'j, is equal to a constant T.

Note: two vectors are **orthogonal** if their inner product is zero.

- $A(B+C) = AB + AC$.
- $(A+B)C = AC + BC$.

- $A(BC) = (AB)C$.

A matrix with elements all zero is called a **null matrix**.

- $(AB)' = B'A'$.
- $(ABC)' = C'B'A'$.

TRACE OF A SQUARE MATRIX

The **trace of a square matrix** A, denoted by tr(A), is defined to be the sum of its diagonal elements.

$$tr(A) = a_{11} + a_{22} + a_{33} + ... + a_{nn}$$

- $tr(A) = A$, if A is a scalar.
- $tr(A') = tr(A)$, because A is square.
- $tr(kA) = k \cdot tr(A)$, where k is a scalar.
- $tr(I_n) = n$, the trace of an identity matrix is its dimension.
- $tr(A \pm B) = tr(A) \pm tr(B)$.
- $tr(AB) = tr(BA)$.
- $tr(AA') = tr(A'A) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2$.

DETERMINANT OF A SQUARE MATRIX

The **determinant of a square matrix** A, denoted by det(A) or $|A|$, is a uniquely defined scalar number associated with the matrix.

i) for a single element matrix (a scalar, $A = a_{11}$), det(A) = a_{11}.

ii) in the (2 x 2) case,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

the determinant is defined to be the difference of two terms as follows,

$$|A| = a_{11}a_{22} - a_{12}a_{21}$$

which is obtained by multiplying the two elements in the principal diagonal of A and then subtracting the product of the two off-diagonal elements.

iii) in the (3 x 3) case,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

iv) for general cases, we start first by defining the **minor** of element a_{ij} as the determinant of the submatrix of A that arises when the i^{th} row and the j^{th} column are deleted and is usually denoted as $|A_{ij}|$. The **cofactor** of the element a_{ij} is $c_{ij} = (-1)^{i+j}|A_{ij}|$. Finally, the determinant of an n x n matrix can be defined as

$$|A| = \sum_{j=1}^{n} a_{ij}c_{ij} \quad \textit{for any row } i = 1,2,\ldots,n.$$

$$= \sum_{i=1}^{n} a_{ij}c_{ij} \quad \textit{for any column } j = 1,2,\ldots,n.$$

- $|A'| = |A|$
- $\begin{vmatrix} a & kc \\ b & kd \end{vmatrix} = \begin{vmatrix} ka & c \\ kb & d \end{vmatrix} = k\begin{vmatrix} a & c \\ b & d \end{vmatrix}$
- $|kA| = k^n|A|$, for scalar k and n x n matrix A.
- If any row (or column) of a matrix is a multiple of any other row (or column) then the determinant is zero, e.g.
  $\begin{vmatrix} a & ka \\ b & kb \end{vmatrix} = k\begin{vmatrix} a & a \\ b & b \end{vmatrix} = k(ab - ab) = 0$
- If A is a diagonal matrix of order n, then $|A| = a_{11}a_{22}\cdots a_{nn}$
- If A and B are square matrices of the same order, then $|AB| = |A||B|$.
- In general, $|A + B| \neq |A| + |B|$

RANK OF A MATRIX AND LINEAR DEPENDENCY

Rank and linear dependency are key concepts for econometrics. The rank of any (m x n) matrix can be defined (i.e., the matrix does not need to be square, as was the case for the determinant and trace) and is inherently linked to the invertibility of the matrix.

The **rank** of a matrix A is equal to the dimension of the largest square submatrix of A that has a nonzero determinant. A matrix is said to be of **rank r** if and only if it has at least one submatrix of order r with a nonzero determinant but has no submatrices of order greater than r with nonzero determinants.

For example, the matrix

$$A = \begin{bmatrix} 4 & 5 & 2 & 14 \\ 3 & 9 & 6 & 21 \\ 8 & 10 & 7 & 28 \\ 1 & 2 & 9 & 5 \end{bmatrix}$$

has rank 3 because $|A| = 0$, but $\begin{vmatrix} 4 & 5 & 2 \\ 3 & 9 & 6 \\ 8 & 10 & 7 \end{vmatrix} = 63 \neq 0$

That is, the largest submatrix of A whose determinant is not zero is of order 3.

The concept of rank also can be viewed in terms of linear dependency. A set of vectors is said to be **linearly dependent** if there is a nontrivial combination (i.e., at least one coefficient in the combination must be nonzero) of the vectors that is equal to the zero vector. More precisely, denote n columns of the matrix A as $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$. This set of these vectors is **linearly dependent** if and only if there exists a set of scalars $\{c_1, c_2, \ldots, c_n\}$, at least one of which is <u>not</u> zero, such that $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \ldots + c_n\mathbf{a}_n = \mathbf{0}$.

In the above example, the columns of the matrix A are linearly dependent because,

$$1\begin{bmatrix} 4 \\ 3 \\ 8 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 9 \\ 10 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 6 \\ 7 \\ 9 \end{bmatrix} - 1\begin{bmatrix} 14 \\ 21 \\ 28 \\ 5 \end{bmatrix} = \mathbf{0}$$

If a set of vectors is not linearly dependent, then it is **linearly independent**. Also, any subset of a linearly independent set of vectors is linearly independent.

In the above example, the first three columns of A are linearly independent, as are the first two columns of A. That is, we cannot find a set of scalars (with at least one nonzero) such that the linear combination of scalars and columns equals the zero vector.

The connection between linear dependency and the rank of a matrix is as follows: **the rank of a matrix A may be defined as the maximum number of linearly independent columns or rows of A.**

In other words, the maximum number of linearly independent columns is equal to the maximum number of linearly independent rows, each being equal to the rank of the matrix. If the maximum number of linearly independent columns (or rows) is equal to the number of columns, then the matrix has **full column rank**. Additionally, if the maximum number of linearly independent rows (or columns) is equal to the number of rows, then the matrix has **full row rank**. When a square matrix A does not have full column/row rank, then its determinant is zero and the matrix is said to be **singular**. When a square matrix A has full row/column rank, its determinant is not zero, and the matrix is said to be **nonsingular** (and therefore invertible).

- Full rank (nonsingular matrix) $\Leftrightarrow$ $|A| \neq 0$ $\Leftrightarrow$ A is invertible.

Furthermore, the maximum number of linearly independent (m x 1) vectors is m. For example, consider the following set of two linearly independent vectors,

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

If there is a third vector,

$$b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

where b_1 *and* b_2 can be any numbers, then the three unknown scalars $c_1, c_2,$ *and* c_3 can always be found by solving the following set of equations,

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In other words, the addition of *any* third vector will result in a (2 x 3) matrix that is not of full rank and therefore not invertible.

Generally speaking, this is because any set of m linearly independent (m x 1) vectors are said to **span** m-dimensional space. This means, by definition, that any (m x 1) vector can be represented as a linear combination of the m vectors that span the space. The set of m vectors therefore is also said to form a **basis** for m-dimensional space.

- $rank(I_n) = n$
- $rank(kA) = rank(A)$, where k is a nonzero constant
- $rank(A') = rank(A)$
- If A is an (m x n) matrix, then $rank(A) \leq \min\{m, n\}$.
- If A and B are matrices, then $rank(AB) \leq \min\{rank(A), rank(B)\}$.
- **If A is an (n x n) matrix, then $rank(A) = n$ if and only if A is nonsingular; $rank(A) < n$ if and only if A is singular.**

There are operations on the rows/columns of a matrix that leave its rank unchanged:

- Multiplication of a row/column of a matrix by a nonzero constant.
- Addition of a scalar multiple of one row/column to another row/column.
- Interchanging two rows/columns.

INVERSE OF A MATRIX

The **inverse of a nonsingular (n x n) matrix** A is another (n x n) matrix, denoted by A^{-1}, that satisfies the following equalities: $A^{-1}A = AA^{-1} = I$. The inverse of a nonsingular (n x n) matrix is unique.

The inverse of a matrix A in terms of its elements can be obtained from the following formula:

- $A^{-1} = \dfrac{C'}{|A|}$ where $C' = [c_{ij}]'$ and $c_{ij} = (-1)^{i+j}|A_{ij}|$

Note that C' is the transpose of the matrix of cofactors of A as defined in the section on determinants. C' is also called the **adjoint** of A.

For example, let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

det(A) = -2 and the cofactors are $c_{11} = 4, c_{22} = 1, c_{12} = -3, c_{21} = -2$. So, the inverse is calculated as,

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}' = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}.$$

- $I^{-1} = I$
- $(A^{-1})^{-1} = A$
- $(A')^{-1} = (A^{-1})'$
- If A is nonsingular, then A^{-1} is nonsingular.
- If A and B are nonsingular, then $(AB)^{-1} = B^{-1}A^{-1}$.

SOLUTIONS FOR SYSTEMS OF SIMULTANEOUS LINEAR EQUATIONS

Consider the following system of linear equations: $Ax = b$ where A is a (m x n) matrix of known coefficients, x is a (n x 1) vector of unknown variables, and b is a (m x 1) vector of known coefficients.

We want to find the conditions under which: 1) the system has no solution, 2) the system has infinitely many solutions, 3) the system has a unique solution. Define the matrix A|b as the augmented matrix of A. This is just the matrix A with the b vector attached on the end. The dimension of A|b is therefore (m x (n+1)).

Succinctly put, the conditions for the three types of solutions are as follows. (Note: there are numerous ways of characterizing the solutions, but we will stick to the simplest representation):

1. The system has *no solution* if rank(A|b) > rank(A).
2. The system has *infinitely many solutions* if rank(A|b) = rank(A) and rank(A) < n.
3. The system has *a unique solution* if rank(A|b) = rank(A) and rank(A) = n.

Let's look at examples for each case.

Case 1: No Solution

Intuition: if rank(A|b) > rank(A), then b is not in the space spanned by A; so b cannot be represented as a linear combination of A; so there is no x that solves (Ax = b); so there is no solution.

Consider the system,

$$\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \end{bmatrix} \qquad or \qquad \begin{aligned} 2x_1 + 3x_2 &= 8 \\ 4x_1 + 6x_2 &= 9 \end{aligned}$$

$$\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 0 \quad \Rightarrow \quad \text{singular}$$

$$rank\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} = 1$$

$$rank\begin{bmatrix} 2 & 3 & 8 \\ 4 & 6 & 9 \end{bmatrix} = 2 \quad \Rightarrow \quad rank(A\,|\,b) > rank(A)$$

If we attempt to solve for x_1 in the first equation and substitute the result into the second equation, the resulting equality is $16 = 9$, which is a contradiction.

<u>Case 2: Infinitely Many Solutions</u>
Intuition: if rank(A|b) = rank(A), then b is in the space spanned by A; so b can be represented as a linear combination of A; so there exists an x that solves (Ax = b). But because rank(A) < n, there are more variables than equations. This gives us “free variables” and therefore multiple solutions, one associated with each choice of values for the free variables.

Consider the following system of equations

$$\begin{bmatrix} 2 & 4 \\ 3 & 6 \\ 4 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \\ 16 \end{bmatrix} \qquad or \qquad \begin{aligned} 2x_1 + 4x_2 &= 8 \\ 3x_1 + 6x_2 &= 12 \\ 4x_1 + 8x_2 &= 16 \end{aligned}$$

$$rank\begin{bmatrix} 2 & 4 \\ 3 & 6 \\ 4 & 8 \end{bmatrix} = 1$$

$$rank\begin{bmatrix} 2 & 4 & 8 \\ 3 & 6 & 12 \\ 4 & 8 & 16 \end{bmatrix} = 1$$

In this case, rank(A|b) = rank(A), but the rank is less than the number of unknown variables (n). Also notice that each equation is just some linear combination of the other two, so we really have only one equation and two unknowns. There are infinitely many solutions that can solve this system, including (4 0)’, (2 1)’, (0 2)’.

Case 3: Unique Solution
Intuition: if rank(A|b) = rank(A), then b is in the space spanned by A; so b can be represented as a linear combination of A; so there exists an x that solves (Ax = b). Because rank(A) = n, there are equal numbers of variables and equations. This gives us no "free variables" and therefore a single solution.

Consider the following system,

$$\begin{bmatrix} 2 & 3 \\ 3 & 5 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ 14 \end{bmatrix} \qquad or \qquad \begin{aligned} 2x_1 + 3x_2 &= 7 \\ 3x_1 + 5x_2 &= 11 \\ 4x_1 + 6x_2 &= 14 \end{aligned}$$

$$rank \begin{bmatrix} 2 & 3 \\ 3 & 5 \\ 4 & 6 \end{bmatrix} = 2$$

$$rank \begin{bmatrix} 2 & 3 & 7 \\ 3 & 5 & 11 \\ 4 & 6 & 14 \end{bmatrix} = 2 \quad because \quad \begin{vmatrix} 2 & 3 & 7 \\ 3 & 5 & 11 \\ 4 & 6 & 14 \end{vmatrix} = 0 \quad and \quad \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 1 \neq 0$$

So, rank(A|b) = rank(A) = 2 = n < m. There is full column rank, and the system can be uniquely solved. In fact, any two independent equations can be used to solve for the x's. The solution is $x_1 = 2, x_2 = 1$.

In econometrics, we often deal with square matrices, so the following is important for us:

- **If A is a square matrix (m = n) and nonsingular, then** $x = A^{-1}b$ **is the unique solution.**

KRONECKER PRODUCT

Let A be an (M x N) matrix and B be a (K x L) matrix. Then the **Kronecker** product (or direct product) of A and B, written as $A \otimes B$, is defined as the (MK x NL) matrix

$$C = A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1N}B \\ a_{21}B & a_{22}B & \cdots & a_{2N}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{M1}B & a_{M2}B & \cdots & a_{MN}B \end{bmatrix}$$

For example if

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

Their Kronecker product is

$$A \otimes B = \begin{bmatrix} 1\begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix} & 3\begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix} \\ 2\begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix} & 0\begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 0 & 6 & 6 & 0 \\ 1 & 0 & 3 & 3 & 0 & 9 \\ 4 & 4 & 0 & 0 & 0 & 0 \\ 2 & 0 & 6 & 0 & 0 & 0 \end{bmatrix}$$

Note that

$$B \otimes A = \begin{bmatrix} 2 & 6 & 2 & 6 & 0 & 0 \\ 4 & 0 & 4 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 3 & 9 \\ 2 & 0 & 0 & 0 & 6 & 0 \end{bmatrix}$$

- $A \otimes B \neq B \otimes A$,
- $(A \otimes B)' = A' \otimes B'$
- $(A \otimes B)(C \otimes D) = AC \otimes BD$
- $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$
- $A \otimes (B + C) = A \otimes B + A \otimes C$

VECTOR AND MATRIX DIFFERENTIATION

In least squares and maximum likelihood estimation, we need to take derivatives of the objective function with respect to a vector of parameters.

Let a function relating y, a scalar, to a set of variables $x_1, x_2, \ldots, x_n$ be $y = f(x_1, x_2, \ldots, x_n)$ or $y = f(\mathbf{x})$, where **x** is an (n x 1) column vector. (Notice that **x** is in bold to indicate a vector.)

The **gradient** of y is the derivatives of y with respect to each element of x as follows

$$\frac{\partial y}{\partial \mathbf{x}} = \begin{bmatrix} \partial y / \partial x_1 \\ \partial y / \partial x_2 \\ \vdots \\ \partial y / \partial x_n \end{bmatrix}$$

Notice the matrix of derivatives of y is a column vector because y is differentiated with respect to x, an (n x 1) column vector.

The same operations can be extended to derivatives of an (m x n) matrix X, such as

$$\frac{\partial y}{\partial X} = \begin{bmatrix} \partial y/\partial x_{11} & \partial y/\partial x_{12} & \cdots & \partial y/\partial x_{1n} \\ \partial y/\partial x_{21} & \partial y/\partial x_{22} & \cdots & \partial y/\partial x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \partial y/\partial x_{m1} & \partial y/\partial x_{m2} & \cdots & \partial y/\partial x_{mn} \end{bmatrix}$$

Notice in this case, the matrix of derivatives is an (m x n) matrix (the same dimension as X).

If, instead, y is an (m x 1) column vector of $y_i, i = 1,2,...,m$ and x is a (n x 1) column vector of $x_j, j = 1,2,...,n$, then the first derivatives of y with respect to x can be represented as an (m x n) matrix, called the **Jacobian** matrix of y with respect to x':

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}'} = \left[\frac{\partial y_i}{\partial x_j}\right] = \begin{bmatrix} \partial y_1/\partial x_1 & \partial y_1/\partial x_2 & \cdots & \partial y_1/\partial x_n \\ \partial y_2/\partial x_1 & \partial y_2/\partial x_2 & \cdots & \partial y_2/\partial x_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial y_m/\partial x_1 & \partial y_m/\partial x_2 & \cdots & \partial y_m/\partial x_n \end{bmatrix}$$

Aside: when differentiating vectors and matrices, note the dimensions of the independent variable (y) and the dependent variables (x). These will determine if the differentiation will entail the transpose of a matrix. In the above example, the first column of the resulting (m x n) matrix is the derivative of the vector of $y_i, i = 1,2,...,m$ with respect to the first x_1. The second column is the derivative with respect to x_2 and so on. Also note that the first row is the derivative of y_1 with respect to the vector x' (a (1 x n) row vector). Therefore because x is a column vector, we need to transpose it to represent the derivative of the m observations of y (down the column) with respect to the n unknown x variables (across the row). The y vector does not need to be transposed because y is represented along the column of the resulting Jacobian matrix.

If we turn back to the scalar case of y, the second derivatives of y with respect to the column vector **x** are defined as follows.

$$\frac{\partial^2 \mathrm{y}}{\partial \mathbf{x} \partial \mathbf{x}'} = \left[\frac{\partial^2 y}{\partial x_i \partial x_j}\right] = \begin{bmatrix} \partial^2 y/\partial x_1^2 & \partial^2 y/\partial x_1 \partial x_2 & \cdots & \partial^2 y/\partial x_1 \partial x_n \\ \partial^2 y/\partial x_2 \partial x_1 & \partial^2 y/\partial x_2^2 & \cdots & \partial^2 y/\partial x_2 \partial x_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial^2 y/\partial x_n \partial x_1 & \partial^2 y/\partial x_n \partial x_2 & \cdots & \partial^2 y/\partial x_n^2 \end{bmatrix}$$

This matrix is symmetric and is called the **Hessian** matrix of y.

Note that the Hessian matrix is just the second derivative of the gradient with respect to the x vector. We need to transpose the x vector when taking the second derivative because for the Hessian, we are taking the derivative of the gradient (a vector) with respect to each x variable. So, the first column is the gradient differentiated with respect to x_1, and the second column is the gradient differentiated with respect to x_2 and so on. So, we need to differentiate the gradient with respect to x' to order these derivatives across the rows of the resulting matrix.

Based on the previous definitions, the rules of derivatives in matrix notation can be established for reference. Consider the following function $z = \mathbf{c}'\mathbf{x}$, where **c** is a (n x 1) vector and does not depend on **x**, and **x** is an (n x 1) vector, and z is a scalar. Then

$$\frac{\partial z}{\partial \mathbf{x}} = \frac{\partial \mathbf{c}'\mathbf{x}}{\partial \mathbf{x}} = \begin{bmatrix} \partial z/\partial x_1 \\ \partial z/\partial x_2 \\ \vdots \\ \partial z/\partial x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \mathbf{c}$$

If $z = C'\mathbf{x}$, where C is an (n x n) matrix and **x** is an (n x 1) vector, then

$$\frac{\partial z'}{\partial \mathbf{x}} = \frac{\partial \mathbf{x}' \mathrm{C}}{\partial \mathbf{x}} = \begin{pmatrix} \mathbf{c_1} & \mathbf{c_2} & \cdots & \mathbf{c_n} \end{pmatrix} = \mathrm{C}$$

where $\mathbf{c}_i$ is the ith column (remember **c** is a vector) of C.

The following formula for the **quadratic form** $z = \mathbf{x}'A\mathbf{x}$ is also useful (<u>for any (n x n) matrix A</u>),

$\frac{\partial z}{\partial \mathbf{x}} = \frac{\partial \mathbf{x}' \mathrm{A}\mathbf{x}}{\partial \mathbf{x}} = A'\mathbf{x} + \mathrm{A}\mathbf{x} = (\mathrm{A}' + \mathrm{A})\mathbf{x}$. The proof of this result is given in the appendix.

If A is a <u>symmetric</u> matrix (A = A'), then

$$\frac{\partial \mathbf{x}' \mathrm{A}\mathbf{x}}{\partial \mathbf{x}} = 2A\mathbf{x}$$

For the second derivatives for any square matrix A,

$$\frac{\partial^2 (\mathbf{x}' \mathrm{A}\mathbf{x})}{\partial \mathbf{x} \partial \mathbf{x}'} = A + \mathrm{A}'$$

and if A = A' (if A is symmetric), then

$$\frac{\partial^2 (\mathbf{x}' \mathrm{A}\mathbf{x})}{\partial \mathbf{x} \partial \mathbf{x}'} = 2\mathrm{A}$$

Some other rules (x is a scalar, unless noted otherwise):

- $\frac{\partial \mathbf{x}' \mathrm{B}\mathbf{y}}{\partial \mathrm{B}} = \mathbf{x}\mathbf{y}'$, where **x** and **y** are (n x 1) column vectors and B is an (n x n) matrix
- $\frac{\partial tr(A)}{\partial \mathrm{A}} = I$

- $\frac{\partial |A|}{\partial \mathrm{A}} = |A|(A')^{-1}$
- $\frac{\partial \ln |A|}{\partial \mathrm{A}} = (A')^{-1}$
- $\frac{\partial AB}{\partial \mathrm{x}} = A\left(\frac{\partial \mathrm{B}}{\partial \mathrm{x}}\right) + \left(\frac{\partial A}{\partial \mathrm{x}}\right)B$
- $\frac{\partial A^{-1}}{\partial \mathrm{x}} = A^{-1}\left(\frac{\partial A}{\partial x}\right)A^{-1}$

Since this review was by no means complete, if you want to learn more about matrix algebra, the following are good references:

Anton, Howard (1994), *Elementary Linear Algebra*, 7th edition, New York: John Wiley & Sons.

The math behind it all. Check out chapters 1, 2, 5.6.

Judge, George G., R. Carter Hill, William E. Griffiths, Helmut Lutkepohl, and Tsoung-Chao Lee (1988), *Introduction to the Theory and Practice of Econometrics*, 2nd Edition, New York: John Wiley & Sons, Appendix A.

These notes follow the Appendix fairly closely.

Leon, Steven J. (1994), *Linear Algebra with Applications*, 4th edition, New Jersey: Prentice Hall.

Simon, Carl P. and Lawrence Blume (1994), *Mathematics for Economists*, New York: W.W. Norton.

Look at chapters 6 – 9, & 26.

APPENDIX

Claim: $\frac{\partial z}{\partial \mathbf{x}} = \frac{\partial \mathbf{x}' \mathrm{A} \mathbf{x}}{\partial \mathbf{x}} = A'\mathbf{x} + \mathrm{A}\mathbf{x} = (\mathrm{A}' + \mathrm{A})\mathbf{x}$

Proof:

Write out the quadratic form for an (n x n) matrix A,

$$z = \mathbf{x}' A \mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{21}x_2 + \ldots + a_{n1}x_n & a_{12}x_1 + a_{22}x_2 + \ldots + a_{n2}x_n & \cdots & a_{1n}x_1 + a_{2n}x_2 + \ldots + a_{nn}x_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \left[x_1 \left(a_{11}x_1 + a_{21}x_2 + \ldots + a_{n1}x_n \right) + x_2 \left(a_{12}x_1 + a_{22}x_2 + \ldots + a_{n2}x_n \right) + \cdots + x_n \left(a_{1n}x_1 + a_{2n}x_2 + \ldots + a_{nn}x_n \right) \right]$$

$$= \begin{bmatrix} a_{11}x_1^2 + a_{22}x_2^2 + \ldots + a_{nn}x_n^2 + (a_{12} + a_{21})x_1x_2 + (a_{13} + a_{31})x_1x_3 + (a_{14} + a_{41})x_1x_4 + \ldots + (a_{1n} + a_{n1})x_1x_n \\ + (a_{23} + a_{32})x_2x_3 + (a_{24} + a_{42})x_2x_4 + \ldots + (a_{2n} + a_{n2})x_2x_n + \ldots + (a_{n,n-1} + a_{n-1,n})x_nx_{n-1} \end{bmatrix}$$

Now differentiate this with respect to the vector x,

$$\frac{\partial z}{\partial \mathbf{x}} = \begin{bmatrix} 2a_{11}x_1 + (a_{21} + a_{12})x_2 + (a_{31} + a_{13})x_3 + \ldots + (a_{n1} + a_{1n})x_n \\ (a_{12} + a_{21})x_1 + 2a_{22}x_2 + (a_{32} + a_{23})x_3 + \ldots + (a_{n2} + a_{2n})x_n \\ \vdots \\ (a_{1n} + a_{n1})x_1 + (a_{2n} + a_{n2})x_2 + (a_{3n} + a_{n3})x_3 + \ldots + 2a_{nn}x_n \end{bmatrix}$$

But this can be rewritten as,

$$\frac{\partial z}{\partial \mathbf{x}} = \begin{bmatrix} 2a_{11} & (a_{21} + a_{12}) & (a_{31} + a_{13}) & \cdots & (a_{n1} + a_{1n}) \\ (a_{12} + a_{21}) & 2a_{22} & (a_{32} + a_{23}) & \ldots & (a_{n2} + a_{2n}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (a_{1n} + a_{n1}) & (a_{2n} + a_{n2}) & (a_{3n} + a_{n3}) & \cdots & 2a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

or

$$\frac{\partial z}{\partial \mathbf{x}} = (A' + A)\mathbf{x}$$

If A is symmetric, then $a_{ij} = a_{ji} \quad for \quad all \quad i, j$, so

$$\frac{\partial z}{\partial \mathbf{x}} = 2A\mathbf{x}$$

This also holds if n = n +1, so, by induction, the result holds for any (n x n) matrix.

Lecture 11: Eigenvalues and Eigenvectors

Definition 11.1. Let A be a square matrix (or linear transformation). A number λ is called an eigenvalue of A if there exists a non-zero vector $\vec{u}$ such that

$$A\vec{u} = \lambda\vec{u}\ . \tag{1}$$

In the above definition, the vector $\vec{u}$ is called an eigenvector associated with this eigenvalue λ. The set of all eigenvectors associated with λ forms a subspace, and is called the eigenspace associated with λ. Geometrically, if we view any $n \times n$ matrix A as a linear transformation T. Then the fact that $\vec{u}$ is an eigenvector associated with an eigenvalue λ means $\vec{u}$ is an invariant direction under T. In other words, the linear transformation T does not change the direction of $\vec{u}$: $\vec{u}$ and $T\vec{u}$ either have the same direction ($\lambda > 0$) or opposite direction ($\lambda < 0$). The eigenvalue is the factor of contraction ($|\lambda| < 1$) or extension ($|\lambda| > 1$).

Remarks. (1) $\vec{u} \neq \vec{0}$ is crucial, since $\vec{u} = \vec{0}$ always satisfies Equ (1). (2) If $\vec{u}$ is an eigenvector for λ, then so is $c\vec{u}$ for any constant c. (3) Geometrically, in 3D, eigenvectors of A are directions that are unchanged under linear transformation A.

We observe from Equ (1) that λ is an eigenvalue iff Equ (1) has a non-trivial solution. Since Equ (1) can be written as

$$(A - \lambda I)\,\vec{u} = A\vec{u} - \lambda\vec{u} = \vec{0}, \tag{2}$$

it follows λ is an eigenvalue iff Equ (2) has a non-trivial solution. By the inverse matrix theorem, Equ (2) has a non-trivial solution iff

$$\det(A - \lambda I) = 0. \tag{3}$$

We conclude that λ is an eigenvalue iff Equ (3) holds. We call Equ (3) "Characteristic Equation" of A. The eigenspace, the subspace of all eigenvectors associated with λ, is

$$\text{eigenspace} = Null\,(A - \lambda I)\,.$$

- **Finding eigenvalues and all independent eigenvectors:**

Step 1. Solve Characteristic Equ (3) for λ.

Step 2. For each λ, find a basis for the eigenspace $Null\,(A - \lambda I)$ (i.e., solution set of Equ (2)).

Example 11.1. Find all eigenvalues and their eigenspace for

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}.$$

Solution:

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{bmatrix}. \end{aligned}$$

The characteristic equation is

$$\det(A - \lambda I) = (3 - \lambda)(-\lambda) - (-2) = 0,$$
$$\lambda^2 - 3\lambda + 2 = 0,$$
$$(\lambda - 1)(\lambda - 2) = 0.$$

We find eigenvalues

$$\lambda_1 = 1, \ \lambda_2 = 2.$$

We next find eigenvectors associated with each eigenvalue. For $\lambda_1 = 1$,

$$\vec{0} = (A - \lambda_1 I)\vec{x} = \begin{bmatrix} 3-1 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

or

$$x_1 = x_2.$$

The parametric vector form of solution set for $(A - \lambda_1 I)\vec{x} = \vec{0}$:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\text{basis of } Null\,(A - I) : \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

This is only (linearly independent) eigenvector for $\lambda_1 = 1$.

The last step can be done slightly differently as follows. From solutions (for $(A - \lambda_1 I)\vec{x} = \vec{0}$)

$$x_1 = x_2,$$

we know there is only one free variable x_2. Therefore, there is only one vector in any basis. To find it, we take x_2 to be any nonzero number, for instance, $x_2 = 1$, and compute $x_1 = x_2 = 1$. We obtain

$$\lambda_1 = 1, \quad \vec{u}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For $\lambda_2 = 2$, we find

$$\vec{0} = (A - \lambda_2 I)\vec{x} = \begin{bmatrix} 3-2 & -2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

or

$$x_1 = 2x_2.$$

To find a basis, we take $x_2 = 1$. Then $x_1 = 2$, and a pair of eigenvalue and eigenvector

$$\lambda_2 = 2, \quad \vec{u}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Example 11.2. Given that 2 is an eigenvalue for

$$A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}.$$

Find a basis of its eigenspace.

Solution:

$$A - 2I = \begin{bmatrix} 4-2 & -1 & 6 \\ 2 & 1-2 & 6 \\ 2 & -1 & 8-2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \to \begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, $(A - 2I)\vec{x} = \vec{0}$ becomes

$$2x_1 - x_2 + 6x_3 = 0, \text{ or } x_2 = 2x_1 + 6x_3, \tag{4}$$

where we select x_1 and x_3 as free variables only to avoid fractions. Solution set in parametric form is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 + 6x_3 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix}.$$

A basis for the eigenspace:

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \text{ and } \vec{u}_2 = \begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix}.$$

Another way of finding a basis for $Null\,(A - \lambda I) = Null\,(A - 2I)$ may be a little easier. From Equ (4), we know that x_1 an x_3 are free variables. Choosing $(x_1,\ x_3) = (1, 0)$ and $(0, 1)$, respectively, we find

$$x_1 = 1,\ x_3 = 0 \Longrightarrow x_2 = 2 \Longrightarrow \vec{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$x_1 = 0,\ x_3 = 1 \Longrightarrow x_2 = 6 \Longrightarrow \vec{u}_2 = \begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix}.$$

Example 11.3. Find eigenvalues: (a)

$$A = \begin{bmatrix} 3 & -1 & 6 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}, \quad A - \lambda I = \begin{bmatrix} 3-\lambda & -1 & 6 \\ 0 & -\lambda & 6 \\ 0 & 0 & 2-\lambda \end{bmatrix}.$$

$$\det(A - \lambda I) = (3 - \lambda)(-\lambda)(2 - \lambda) = 0$$

The eigenvalues are 3, 0, 2, exactly the diagonal elements. (b)

$$B = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 4 \end{bmatrix}, \quad B - \lambda I = \begin{bmatrix} 4-\lambda & 0 & 0 \\ 2 & 1-\lambda & 0 \\ 1 & 0 & 4-\lambda \end{bmatrix}$$

$$\det(B - \lambda I) = (4-\lambda)^2 (1-\lambda) = 0.$$

The eigenvalues are 4, 1, 4 (4 is a double root), exactly the diagonal elements.

Theorem 11.1. (1) The eigenvalues of a triangle matrix are its diagonal elements.

(2) Eigenvectors for different eigenvalues are linearly independent. More precisely, suppose that λ_1, $\lambda_2, ...,$ λ_p are p different eigenvalues of a matrix A. Then, the set consisting of

$$\text{a basis of } Null\,(A - \lambda_1 I)\,, \text{ a basis of } Null\,(A - \lambda_2 I)\,, ..., \text{ a basis of } Null\,(A - \lambda_p I)$$

is linearly independent.

Proof. (2) For simplicity, we assume $p = 2 : \lambda_1 \neq \lambda_2$ are two different eigenvalues. Suppose that $\vec{u}_1$ is an eigenvector of λ_1 and $\vec{u}_2$ is an eigenvector of λ_2 To show independence, we need to show that the only solution to

$$x_1\vec{u}_1 + x_2\vec{u}_2 = \vec{0}$$

is $x_1 = x_2 = 0$. Indeed, if $x_1 \neq 0$, then

$$\vec{u}_1 = \frac{x_2}{x_1}\vec{u}_2. \tag{5}$$

We now apply A to the above equation. It leads to

$$A\vec{u}_1 = \frac{x_2}{x_1}A\vec{u}_2 \Longrightarrow \lambda_1\vec{u}_1 = \frac{x_2}{x_1}\lambda_2\vec{u}_2. \tag{6}$$

Equ (5) and Equ (6) are contradictory to each other: by Equ (5),

$$\begin{aligned} \text{Equ (5)} &\Longrightarrow \lambda_1\vec{u}_1 = \frac{x_2}{x_1}\lambda_1\vec{u}_2 \\ \text{Equ (6)} &\Longrightarrow \lambda_1\vec{u}_1 = \frac{x_2}{x_1}\lambda_2\vec{u}_2, \end{aligned}$$

or

$$\frac{x_2}{x_1}\lambda_1\vec{u}_2 = \lambda_1\vec{u}_1 = \frac{x_2}{x_1}\lambda_2\vec{u}_2.$$

Therefor $\lambda_1 = \lambda_2$, a contradiction to the assumption that they are different eigenvalues. ■

- Characteristic Polynomials

We know that the key to find eigenvalues and eigenvectors is to solve the Characteristic Equation (3)

$$\det(A - \lambda I) = 0.$$

For 2×2 matrix,

$$A - \lambda I = \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix}.$$

So

$$\begin{aligned} \det(A - \lambda I) &= (a - \lambda)(d - \lambda) - bc \\ &= \lambda^2 + (-a - d)\lambda + (ad - bc) \end{aligned}$$

is a quadratic function (i.e., a polynomial of degree 2). In general, for any $n \times n$ matrix A,

$$A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix}.$$

We may expand the determinant along the first row to get

$$\det(A - \lambda I) = (a_{11} - \lambda)\det \begin{bmatrix} a_{22} - \lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix} + \ldots$$

By induction, we see that $\det(A - \lambda I)$ is a polynomial of degree n.We called this polynomial the **characteristic polynomial** of A. Consequently, there are total of n (the number of rows in the matrix A) eigenvalues (real or complex, after taking account for multiplicity). Finding roots for higher order polynomials may be very challenging.

Example 11.4. Find all eigenvalues for

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Solution:

$$A - \lambda I = \begin{bmatrix} 5 - \lambda & -2 & 6 & -1 \\ 0 & 3 - \lambda & -8 & 0 \\ 0 & 0 & 5 - \lambda & 4 \\ 0 & 0 & 1 & 1 - \lambda \end{bmatrix},$$

$$\begin{aligned} \det(A - \lambda I) &= (5 - \lambda)\det \begin{bmatrix} 3 - \lambda & -8 & 0 \\ 0 & 5 - \lambda & 4 \\ 0 & 1 & 1 - \lambda \end{bmatrix} \\ &= (5 - \lambda)(3 - \lambda)\det \begin{bmatrix} 5 - \lambda & 4 \\ 1 & 1 - \lambda \end{bmatrix} \\ &= (5 - \lambda)(3 - \lambda)\left[(5 - \lambda)(1 - \lambda) - 4\right] = 0. \end{aligned}$$

There are 4 roots:

$$\begin{aligned}(5-\lambda) &= 0 \implies \lambda = 5\\ (3-\lambda) &= 0 \implies \lambda = 3\\ (5-\lambda)(1-\lambda)-4 &= 0 \implies \lambda^2 - 6\lambda + 1 = 0\\ &\implies \lambda = \frac{6 \pm \sqrt{36-4}}{2} = 3 \pm 2\sqrt{2}.\end{aligned}$$

We know that we can computer determinants using elementary row operations. One may ask: Can we use elementary row operations to find eigenvalues? More specifically, we have

Question: Suppose that B is obtained from A by elementary row operations. Do A and B has the same eigenvalues? (Ans: No)

Example 11.5.

$$A = \begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix} \overset{R_2+R_1 \to R_2}{\to} \begin{bmatrix}1 & 1\\ 1 & 3\end{bmatrix} = B$$

A has eigenvalues 1 and 2. For B, the characteristic equation is

$$\begin{aligned}\det(B-\lambda I) &= \begin{bmatrix}1-\lambda & 1\\ 1 & 3-\lambda\end{bmatrix}\\ &= (1-\lambda)(3-\lambda) - 1 = \lambda^2 - 4\lambda + 2.\end{aligned}$$

The eigenvalues are

$$\lambda = \frac{4 \pm \sqrt{16-8}}{2} = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}.$$

This example show that row operation may completely change eigenvalues.

Definition 11.2. Two $n \times n$ matrices A and B are called similar, and is denoted as $A \sim B$, if there exists an invertible matrix P such that $A = PBP^{-1}$.

Theorem 11.2. If A and B are similar, then they have exact the same characteristic polynomial and consequently the same eigenvalues.

Indeed, if $A = PBP^{-1}$, then $P(B-\lambda I)P^{-1} = PBP^{-1} - \lambda PIP^{-1} = (A-\lambda I)$. Therefore,

$$\det(A-\lambda I) = \det\left(P(B-\lambda I)P^{-1}\right) = \det(P)\det(B-\lambda I)\det\left(P^{-1}\right) = \det(B-\lambda I).$$

Example 11.6. Find eigenvalues of A if

$$A \sim B = \begin{bmatrix}5 & -2 & 6 & -1\\ 0 & 3 & -8 & 0\\ 0 & 0 & 5 & 4\\ 0 & 0 & 0 & 4\end{bmatrix}.$$

Solution: Eigenvalues of B are $\lambda = 5, 3, 5, 4$. These are also the eigenvalues of A.

Caution: If $A \sim B$, and if λ_0 is an eigenvalue for A and B, then an corresponding eigenvector for A may not be an eigenvector for B. In other words, two similar matrices A and B have the same eigenvalues but different eigenvectors.

Example 11.7. Though row operation alone will not preserve eigenvalues, a pair of row and column operation do maintain similarity. We first observe that if P is a type 1 elementary matrix (row replacement) ,

$$P = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \overset{aR_1+R_2\to R_2}{\longleftarrow} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

then its inverse P^{-1} is a type 1 (column) elementary matrix obtained from the identity matrix by an elementary column operation that is of the same kind with "opposite sign" to the previous row operation, i.e.,

$$P^{-1} = \begin{bmatrix} 1 & 0 \\ -a & 1 \end{bmatrix} \overset{C_1-aC_2\to C_1}{\longleftarrow} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We call the column operation

$$C_1 - aC_2 \to C_1$$

is "inverse" to the row operation

$$R_1 + aR_2 \to R_2.$$

Now we perform a row operation on A followed immediately by the column operation inverse to the row operation

$$\begin{aligned} A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} &\overset{R_1+R_2\to R_2}{\to} \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \text{ (left multiply by } P) \\ \overset{C_1-C_2\to C_1}{\to} &\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} = B \text{ (right multiply by } P^{-1}.) \end{aligned}$$

We can verify that A and B are similar through P (with $a = 1$)

$$\begin{aligned} PAP^{-1} &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}. \end{aligned}$$

Now, $\lambda_1 = 1$ is an eigenvalue. Then,

$$\begin{aligned} (A-1)\vec{u} &= \begin{bmatrix} 1-1 & 1 \\ 0 & 2-1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &\Longrightarrow \vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ is an eigenvector for } A. \end{aligned}$$

But

$$
\begin{aligned}
(B-1)\,\vec{u} &= \begin{bmatrix} 0-1 & 1 \\ -2 & 3-1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \\
&\Longrightarrow \vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ is NOT an eigenvector for } B.
\end{aligned}
$$

In fact,

$$
\begin{aligned}
(B-1)\,\vec{v} &= \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \\
\text{So, } \vec{v} &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ is an eigenvector for } B.
\end{aligned}
$$

This example shows that

1. Row operation alone will not preserve eigenvalues.
2. Two similar matrices share the same characteristics polynomial and same eigenvalues. But they have different eigenvectors.

- **Homework #11.**

1. Find eigenvalues if

   (a) $A \sim \begin{bmatrix} -1 & 2 & 8 & -1 \\ 0 & 2 & 10 & 0 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$

   (b) $B \sim \begin{bmatrix} -1 & 2 & 8 & -1 \\ 1 & 2 & 10 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

2. Find eigenvalues and a basis of each eigenspace.

   (a) $A = \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix}.$

   (b) $B = \begin{bmatrix} 7 & 4 & 6 \\ -3 & -1 & -8 \\ 0 & 0 & 1 \end{bmatrix}.$

3. Find a basis of the eigenspace associated with eigenvalue $\lambda = 1$ for

$$A = \begin{bmatrix} 1 & 2 & 4 & -1 \\ 1 & 2 & -3 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

4. Determine true or false. Reason your answers.

   (a) If $A\vec{x} = \lambda\vec{x}$, then λ is an eigenvalue of A.

   (b) A is invertible iff 0 is not an eigenvalue.

   (c) If $A \sim B$, then A and B has the same eigenvalues and eigenspaces.

   (d) If A and B have the same eigenvalues, then they have the same characteristic polynomial.

   (e) If $\det A = \det B$, then A is similar to B.

1.6 Vector Calculus 1 - Differentiation

Calculus involving vectors is discussed in this section, rather intuitively at first and more formally toward the end of this section.

1.6.1 The Ordinary Calculus

Consider a **scalar-valued function of a scalar**, for example the time-dependent density of a material $\rho = \rho(t)$. The calculus of scalar valued functions of scalars is just the ordinary calculus. Some of the important concepts of the ordinary calculus are reviewed in Appendix B to this Chapter, §1.B.2.

1.6.2 Vector-valued Functions of a scalar

Consider a **vector-valued function of a scalar**, for example the time-dependent displacement of a particle $\mathbf{u} = \mathbf{u}(t)$. In this case, the derivative is defined in the usual way,

$$\frac{d\mathbf{u}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{u}(t + \Delta t) - \mathbf{u}(t)}{\Delta t},$$

which turns out to be simply the derivative of the coefficients[1],

$$\frac{d\mathbf{u}}{dt} = \frac{du_1}{dt}\mathbf{e}_1 + \frac{du_2}{dt}\mathbf{e}_2 + \frac{du_3}{dt}\mathbf{e}_3 \equiv \frac{du_i}{dt}\mathbf{e}_i$$

Partial derivatives can also be defined in the usual way. For example, if **u** is a function of the coordinates, $\mathbf{u}(x_1, x_2, x_3)$, then

$$\frac{\partial \mathbf{u}}{\partial x_1} = \lim_{\Delta x_1 \to 0} \frac{\mathbf{u}(x_1 + \Delta x_1, x_2, x_3) - \mathbf{u}(x_1, x_2, x_3)}{\Delta x_1}$$

Differentials of vectors are also defined in the usual way, so that when u_1, u_2, u_3 undergo increments $du_1 = \Delta u_1$, $du_2 = \Delta u_2$, $du_3 = \Delta u_3$, the differential of **u** is

$$d\mathbf{u} = du_1\mathbf{e}_1 + du_2\mathbf{e}_2 + du_3\mathbf{e}_3$$

and the differential and actual increment $\Delta\mathbf{u}$ approach one another as $\Delta u_1, \Delta u_2, \Delta u_3 \to 0$.

[1] assuming that the base vectors do not depend on t

Space Curves

The derivative of a vector can be interpreted geometrically as shown in Fig. 1.6.1: $\Delta\mathbf{u}$ is the increment in **u** consequent upon an increment Δt in *t*. As *t* changes, the end-point of the vector $\mathbf{u}(t)$ traces out the dotted curve Γ shown – it is clear that as $\Delta t \to 0$, $\Delta\mathbf{u}$ approaches the tangent to Γ, so that $d\mathbf{u}/dt$ is tangential to Γ. The unit vector tangent to the curve is denoted by $\boldsymbol{\tau}$:

$$\boldsymbol{\tau} = \frac{d\mathbf{u}/dt}{|d\mathbf{u}/dt|} \tag{1.6.1}$$

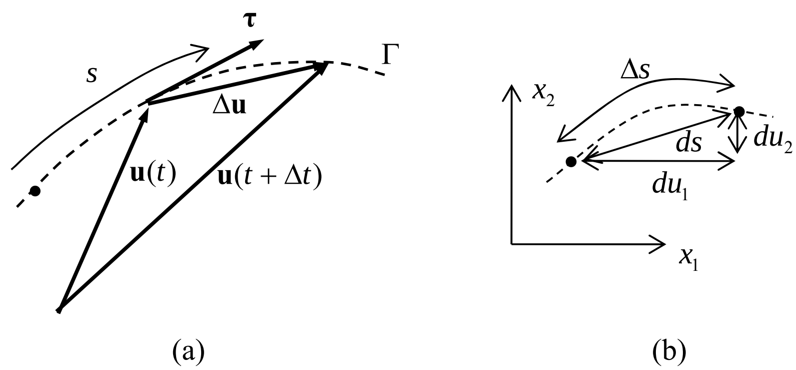


Figure 1.6.1: a space curve; (a) the tangent vector, (b) increment in arc length

Let *s* be a measure of the length of the curve Γ, measured from some fixed point on Γ. Let Δs be the increment in arc-length corresponding to increments in the coordinates, $\Delta\mathbf{u} = [\Delta u_1, \Delta u_2, \Delta u_3]^{\mathrm{T}}$, Fig. 1.6.1b. Then, from the ordinary calculus (see Appendix 1.B),

$$(ds)^2 = (du_1)^2 + (du_2)^2 + (du_3)^2$$

so that

$$\frac{ds}{dt} = \sqrt{\left(\frac{du_1}{dt}\right)^2 + \left(\frac{du_2}{dt}\right)^2 + \left(\frac{du_3}{dt}\right)^2}$$

But

$$\frac{d\mathbf{u}}{dt} = \frac{du_1}{dt}\mathbf{e}_1 + \frac{du_2}{dt}\mathbf{e}_2 + \frac{du_3}{dt}\mathbf{e}_3$$

so that

$$\left|\frac{d\mathbf{u}}{dt}\right| = \frac{ds}{dt} \tag{1.6.2}$$

Thus the unit vector tangent to the curve can be written as

$$\boldsymbol{\tau} = \frac{d\mathbf{u}/dt}{ds/dt} = \frac{d\mathbf{u}}{ds} \tag{1.6.3}$$

If **u** is interpreted as the position vector of a particle and t is interpreted as time, then $\mathbf{v} = d\mathbf{u}/dt$ is the velocity vector of the particle as it moves with speed ds/dt along Γ.

Example (of particle motion)

A particle moves along a curve whose parametric equations are $x_1 = 2t^2$, $x_2 = t^2 - 4t$, $x_3 = 3t - 5$ where t is time. Find the component of the velocity at time $t = 1$ in the direction $\mathbf{a} = \mathbf{e}_1 - 3\mathbf{e}_2 + 2\mathbf{e}_3$.

Solution

The velocity is

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{r}}{dt} = \frac{d}{dt}\left\{2t^2\mathbf{e}_1 + \left(t^2 - 4t\right)\mathbf{e}_2 + \left(3t - 5\right)\mathbf{e}_3\right\} \\ &= 4\mathbf{e}_1 - 2\mathbf{e}_2 + 3\mathbf{e}_3 \quad \text{at } t = 1\end{aligned}$$

The component in the given direction is $\mathbf{v} \cdot \hat{\mathbf{a}}$, where $\hat{\mathbf{a}}$ is a unit vector in the direction of **a**, giving $8\sqrt{14}/7$.

■

Curvature

The scalar **curvature** $\kappa(s)$ of a space curve is defined to be the magnitude of the rate of change of the unit tangent vector:

$$\kappa(s) = \left|\frac{d\boldsymbol{\tau}}{ds}\right| = \left|\frac{d^2\mathbf{u}}{ds^2}\right| \tag{1.6.4}$$

Note that $\Delta\boldsymbol{\tau}$ is in a direction perpendicular to $\boldsymbol{\tau}$, Fig. 1.6.2. In fact, this can be proved as follows: since $\boldsymbol{\tau}$ is a unit vector, $\boldsymbol{\tau} \cdot \boldsymbol{\tau}$ is a constant ($= 1$), and so $d(\boldsymbol{\tau} \cdot \boldsymbol{\tau})/ds = 0$, but also,

$$\frac{d}{ds}(\boldsymbol{\tau} \cdot \boldsymbol{\tau}) = 2\boldsymbol{\tau} \cdot \frac{d\boldsymbol{\tau}}{ds}$$

and so $\boldsymbol{\tau}$ and $d\boldsymbol{\tau}/ds$ are perpendicular. The unit vector defined in this way is called the **principal normal vector**:

$$\mathbf{v} = \frac{1}{\kappa}\frac{d\boldsymbol{\tau}}{ds} \tag{1.6.5}$$

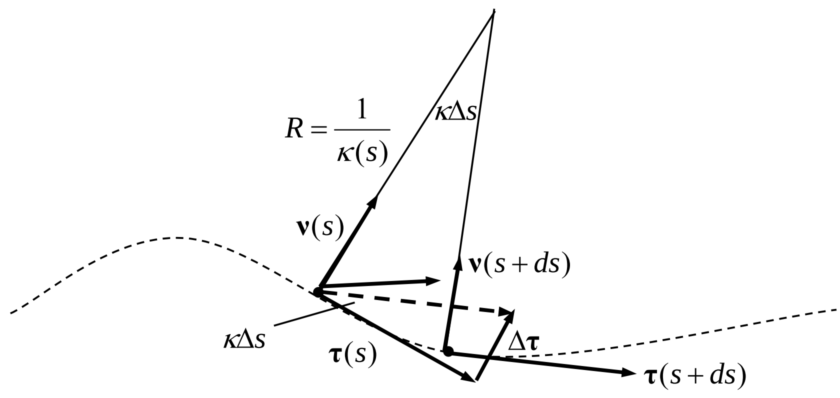


Figure 1.6.2: the curvature

This can be seen geometrically in Fig. 1.6.2: from Eqn. 1.6.5, $\Delta\boldsymbol{\tau}$ is a vector of magnitude $\kappa\Delta s$ in the direction of the vector normal to $\boldsymbol{\tau}$. The **radius of curvature** *R* is defined as the reciprocal of the curvature; it is the radius of the circle which just touches the curve at *s*, Fig. 1.6.2.

Finally, the unit vector perpendicular to both the tangent vector and the principal normal vector is called the **unit binormal vector**:

$$\mathbf{b} = \boldsymbol{\tau} \times \mathbf{v} \tag{1.6.6}$$

The planes defined by these vectors are shown in Fig. 1.6.3; they are called the **rectifying plane**, the **normal plane** and the **osculating plane**.

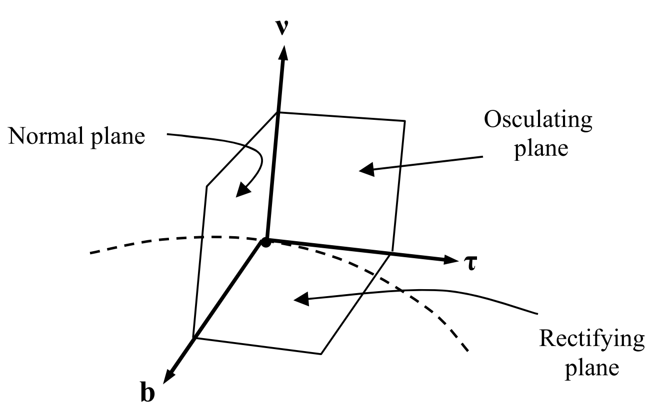


Figure 1.6.3: the unit tangent, principal normal and binormal vectors and associated planes

Rules of Differentiation

The derivative of a vector is also a vector and the usual rules of differentiation apply,

$$\frac{d}{dt}(\mathbf{u}+\mathbf{v}) = \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{v}}{dt}$$
$$\frac{d}{dt}(\alpha(t)\mathbf{v}) = \alpha\frac{d\mathbf{v}}{dt} + \mathbf{v}\frac{d\alpha}{dt} \tag{1.6.7}$$

Also, it is straight forward to show that {▲Problem 2}

$$\frac{d}{dt}(\mathbf{v}\cdot\mathbf{a}) = \mathbf{v}\cdot\frac{d\mathbf{a}}{dt} + \frac{d\mathbf{v}}{dt}\cdot\mathbf{a} \qquad \frac{d}{dt}(\mathbf{v}\times\mathbf{a}) = \mathbf{v}\times\frac{d\mathbf{a}}{dt} + \frac{d\mathbf{v}}{dt}\times\mathbf{a} \tag{1.6.8}$$

(The order of the terms in the cross-product expression is important here.)

1.6.3 Fields

In many applications of vector calculus, a scalar or vector can be associated with each point in space $\mathbf{x}$. In this case they are called **scalar** or **vector fields**. For example

$\theta(\mathbf{x})$	temperature	a scalar field (a scalar-valued function of position)
$\mathbf{v}(\mathbf{x})$	velocity	a vector field (a vector valued function of position)

These quantities will in general depend also on time, so that one writes $\theta(\mathbf{x},t)$ or $\mathbf{v}(\mathbf{x},t)$. Partial differentiation of scalar and vector fields with respect to the variable t is symbolised by $\partial/\partial t$. On the other hand, partial differentiation with respect to the coordinates is symbolised by $\partial/\partial x_i$. The notation can be made more compact by introducing the **subscript comma** to denote partial differentiation with respect to the coordinate variables, in which case $\phi_{,i} = \partial\phi/\partial x_i$, $u_{i,jk} = \partial^2 u_i/\partial x_j \partial x_k$, and so on.

1.6.4 The Gradient of a Scalar Field

Let $\phi(\mathbf{x})$ be a scalar field. The **gradient** of ϕ is a vector field defined by (see Fig. 1.6.4)

$$\begin{aligned}\nabla\phi &= \frac{\partial\phi}{\partial x_1}\mathbf{e}_1 + \frac{\partial\phi}{\partial x_2}\mathbf{e}_2 + \frac{\partial\phi}{\partial x_3}\mathbf{e}_3 \\ &= \frac{\partial\phi}{\partial x_i}\mathbf{e}_i \\ &\equiv \frac{\partial\phi}{\partial\mathbf{x}}\end{aligned} \qquad \textbf{Gradient of a Scalar Field} \tag{1.6.9}$$

The gradient $\nabla\phi$ is of considerable importance because if one takes the dot product of $\nabla\phi$ with $d\mathbf{x}$, it gives the increment in ϕ:

$$\begin{aligned}\nabla\phi\cdot d\mathbf{x} &= \frac{\partial\phi}{\partial x_i}\mathbf{e}_i\cdot dx_j\mathbf{e}_j \\ &= \frac{\partial\phi}{\partial x_i}dx_i \\ &= d\phi \\ &= \phi(\mathbf{x}+d\mathbf{x})-\phi(d\mathbf{x})\end{aligned} \tag{1.6.10}$$

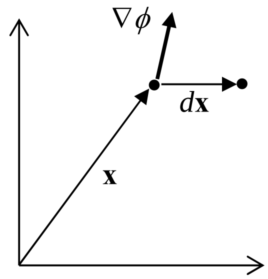


Figure 1.6.4: the gradient of a vector

If one writes $d\mathbf{x}$ as $|d\mathbf{x}|\mathbf{e} = dx\mathbf{e}$, where **e** is a unit vector in the direction of $d\mathbf{x}$, then

$$\nabla\phi\cdot\mathbf{e} = \left(\frac{d\phi}{dx}\right)_{\text{in } \mathbf{e} \text{ direction}} \equiv \frac{d\phi}{dn} \tag{1.6.11}$$

This quantity is called the **directional derivative** of ϕ, in the direction of **e**, and will be discussed further in §1.6.11.

The gradient of a scalar field is also called the **scalar gradient**, to distinguish it from the **vector gradient** (see later)[2], and is also denoted by

$$\text{grad}\,\phi \equiv \nabla\phi \tag{1.6.12}$$

Example (of the Gradient of a Scalar Field)

Consider a two-dimensional temperature field $\theta = x_1^2 + x_2^2$. Then

$$\nabla\theta = 2x_1\mathbf{e}_1 + 2x_2\mathbf{e}_2$$

For example, at $(1,0)$, $\theta = 1$, $\nabla\theta = 2\mathbf{e}_1$ and at $(1,1)$, $\theta = 2$, $\nabla\theta = 2\mathbf{e}_1 + 2\mathbf{e}_2$, Fig. 1.6.5. Note the following:

(i) $\nabla\theta$ points in the direction *normal* to the curve $\theta = \text{const}$.
(ii) the direction of *maximum* rate of change of θ is in the direction of $\nabla\theta$

[2] in this context, a *gradient* is a derivative with respect to a position vector, but the term gradient is used more generally than this, e.g. see §1.14

(iii) the direction of zero $d\theta$ is in the direction *perpendicular* to $\nabla\theta$

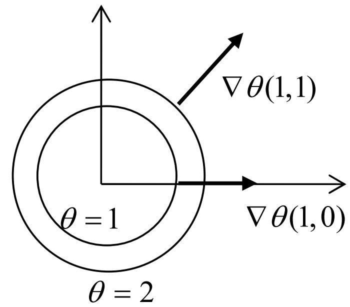


Figure 1.6.5: gradient of a temperature field

The curves $\theta(x_1, x_2) = \text{const.}$ are called **isotherms** (curves of constant temperature). In general, they are called **iso-curves** (or **iso-surfaces** in three dimensions).

■

Many physical laws are given in terms of the gradient of a scalar field. For example, **Fourier's law** of heat conduction relates the heat flux **q** (the rate at which heat flows through a surface of unit area[3]) to the temperature gradient through

$$\mathbf{q} = -k\nabla\theta \tag{1.6.13}$$

where *k* is the **thermal conductivity** of the material, so that heat flows along the direction normal to the isotherms.

The Normal to a Surface

In the above example, it was seen that $\nabla\theta$ points in the direction normal to the curve $\theta = \text{const.}$ Here it will be seen generally how and why the gradient can be used to obtain a normal vector to a surface.

Consider a surface represented by the scalar function $f(x_1, x_2, x_3) = c$, c a constant[4], and also a space curve *C* lying on the surface, defined by the position vector $\mathbf{r} = x_1(t)\mathbf{e}_1 + x_2(t)\mathbf{e}_2 + x_3(t)\mathbf{e}_3$. The components of **r** must satisfy the equation of the surface, so $f(x_1(t), x_2(t), x_3(t)) = c$. Differentiation gives

$$\frac{df}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial f}{\partial x_3}\frac{dx_3}{dt} = 0$$

[3] the **flux** is the rate of flow of fluid, particles or energy through a given surface; the **flux density** is the flux per unit area but, as here, this is more commonly referred to simply as the flux

[4] a surface can be represented by the equation $f(x_1, x_2, x_3) = c$; for example, the expression $x_1^2 + x_2^2 + x_3^2 = 4$ is the equation for a sphere of radius 2 (with centre at the origin). Alternatively, the surface can be written in the form $x_3 = g(x_1, x_2)$, for example $x_3 = \sqrt{4 - x_1^2 - x_2^2}$

which is equivalent to the equation $\operatorname{grad} f \cdot (d\mathbf{r}/dt) = 0$ and, as seen in §1.6.2, $d\mathbf{r}/dt$ is a vector tangential to the surface. Thus $\operatorname{grad} f$ is normal to the tangent vector; $\operatorname{grad} f$ must be normal to all the tangents to all the curves through *p*, so it must be normal to the plane tangent to the surface.

Taylor's Series

Writing ϕ as a function of three variables (omitting time *t*), so that $\phi = \phi(x_1, x_2, x_3)$, then ϕ can be expanded in a three-dimensional Taylor's series:

$$\phi(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3) = \phi(x_1, x_2, x_3) + \left\{ \frac{\partial \phi}{\partial x_1} dx_1 + \frac{\partial \phi}{\partial x_2} dx_2 + \frac{\partial \phi}{\partial x_3} dx_3 \right\} + \frac{1}{2} \left\{ \frac{\partial^2 \phi}{\partial x_1^2} (dx_1)^2 + \cdots \right\}$$

Neglecting the higher order terms, this can be written as

$$\phi(\mathbf{x} + d\mathbf{x}) = \phi(\mathbf{x}) + \frac{\partial \phi}{\partial \mathbf{x}} \cdot d\mathbf{x}$$

which is equivalent to 1.6.9, 1.6.10.

1.6.5 The Nabla Operator

The symbolic vector operator ∇ is called the **Nabla operator**[5]. One can write this in component form as

$$\nabla = \mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \mathbf{e}_3 \frac{\partial}{\partial x_3} = \mathbf{e}_i \frac{\partial}{\partial x_i} \tag{1.6.14}$$

One can generalise the idea of the gradient of a scalar field by defining the dot product and the cross product of the vector operator ∇ with a vector field $(\bullet)$, according to the rules

$$\nabla \cdot (\bullet) = \mathbf{e}_i \frac{\partial}{\partial x_i} \cdot (\bullet), \quad \nabla \times (\bullet) = \mathbf{e}_i \frac{\partial}{\partial x_i} \times (\bullet) \tag{1.6.15}$$

The following terminology is used:

$$\begin{aligned} \operatorname{grad} \phi &= \nabla \phi \\ \operatorname{div} \mathbf{u} &= \nabla \cdot \mathbf{u} \\ \operatorname{curl} \mathbf{u} &= \nabla \times \mathbf{u} \end{aligned} \tag{1.6.16}$$

[5] or **del** or the **Gradient operator**

These latter two are discussed in the following sections.

1.6.6 The Divergence of a Vector Field

From the definition (1.6.15), the **divergence** of a vector field $\mathbf{a}(\mathbf{x})$ is the scalar field

$$\begin{aligned}\operatorname{div}\mathbf{a} = \nabla\cdot\mathbf{a} &= \left(\mathbf{e}_i \frac{\partial}{\partial x_i}\right)\cdot\left(a_j\mathbf{e}_j\right) = \frac{\partial a_i}{\partial x_i} \\ &= \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}\end{aligned}$$

Divergence of a Vector Field (1.6.17)

Differential Elements & Physical interpretations of the Divergence

Consider a flowing compressible[6] material with velocity field $\mathbf{v}(x_1, x_2, x_3)$. Consider now a **differential element** of this material, with dimensions $\Delta x_1, \Delta x_2, \Delta x_3$, with bottom left-hand corner at (x_1, x_2, x_3), fixed in space and through which the material flows[7], Fig. 1.6.6.

The component of the velocity in the x_1 direction, v_1, will vary over a face of the element but, *if the element is small*, the velocities will vary linearly as shown; only the components at the four corners of the face are shown for clarity.

Since [distance = time × velocity], the volume of material flowing through the right-hand face in time Δt is Δt times the "volume" bounded by the four corner velocities (between the right-hand face and the plane surface denoted by the dotted lines); it is straightforward to show that this volume is equal to the volume shown to the right, Fig. 1.6.6b, with constant velocity equal to the average velocity v_{ave}, which occurs at the centre of the face. Thus the volume of material flowing out is[8] $\Delta x_2 \Delta x_3 v_{ave} \Delta t$ and the **volume flux**, i.e. the *rate* of volume flow, is $\Delta x_2 \Delta x_3 v_{ave}$. Now

$$v_{ave} = v_1\left(x_1 + \Delta x_1, x_2 + \tfrac{1}{2}\Delta x_2, x_3 + \tfrac{1}{2}\Delta x_3\right)$$

Using a Taylor's series expansion, and neglecting higher order terms,

$$v_{ave} \approx v_1(x_1, x_2, x_3) + \Delta x_1 \frac{\partial v_1}{\partial x_1} + \tfrac{1}{2}\Delta x_2 \frac{\partial v_1}{\partial x_2} + \tfrac{1}{2}\Delta x_3 \frac{\partial v_1}{\partial x_3}$$

---

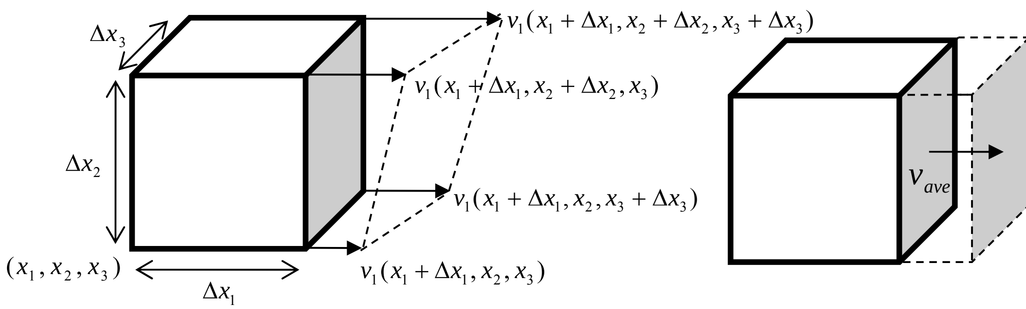
[6] that is, it can be compressed or expanded

[7] this type of fixed volume in space, used in analysis, is called a **control volume**

[8] the velocity will change by a small amount during the time interval Δt. One could use the average velocity in the calculation, i.e. $\frac{1}{2}\left(v_1(\mathbf{x}, t) + v_1(\mathbf{x}, t + \Delta t)\right)$, but in the limit as $\Delta t \to 0$, this will reduce to $v_1(\mathbf{x}, t)$

with the partial derivatives evaluated at (x_1, x_2, x_3), so the volume flux out is

$$\Delta x_2 \Delta x_3 \left\{ v_1(x_1, x_2, x_3) + \Delta x_1 \frac{\partial v_1}{\partial x_1} + \tfrac{1}{2}\Delta x_2 \frac{\partial v_1}{\partial x_2} + \tfrac{1}{2}\Delta x_3 \frac{\partial v_1}{\partial x_3} \right\}$$



(a) (b)

Figure 1.6.6: a differential element; (a) flow through a face, (b) volume of material flowing through the face

The net volume flux out (rate of volume flow out through the right-hand face minus the rate of volume flow in through the left-hand face) is then $\Delta x_1 \Delta x_2 \Delta x_3 (\partial v_1 / \partial x_1)$ and the net volume flux per unit volume is $\partial v_1 / \partial x_1$. Carrying out a similar calculation for the other two coordinate directions leads to

$$\textit{net unit volume flux out of an elemental volume}: \quad \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \equiv \operatorname{div}\mathbf{v} \qquad (1.6.18)$$

which is the physical meaning of the divergence of the velocity field.

If $\operatorname{div}\mathbf{v} > 0$, there is a net flow out and the density of material is decreasing. On the other hand, if $\operatorname{div}\mathbf{v} = 0$, the inflow equals the outflow and the density remains constant – such a material is called **incompressible**[9]. A flow which is divergence free is said to be **isochoric**. A vector $\mathbf{v}$ for which $\operatorname{div}\mathbf{v} = 0$ is said to be **solenoidal**.

Notes:

- The above result holds only in the limit when the element shrinks to zero size – so that the extra terms in the Taylor series tend to zero and the velocity field varies in a linear fashion over a face
- consider the velocity at a fixed point in space, $\mathbf{v}(\mathbf{x}, t)$. The velocity at a later time, $\mathbf{v}(\mathbf{x}, t + \Delta t)$, actually gives the velocity of a different material particle. This is shown in Fig. 1.6.7 below: the material particles $1, 2, 3$ are moving through space and whereas $\mathbf{v}(\mathbf{x}, t)$ represents the velocity of particle 2, $\mathbf{v}(\mathbf{x}, t + \Delta t)$ now represents the velocity of particle 1, which has moved into position $\mathbf{x}$. This point is important in the consideration of the kinematics of materials, to be discussed in Chapter 2

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[9] a **liquid**, such as water, is a material which is incompressible

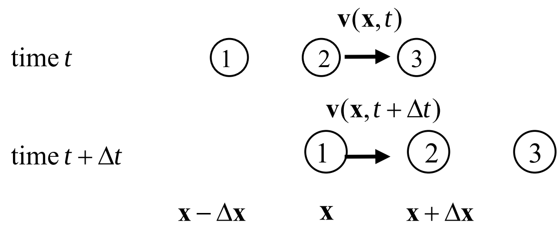


Figure 1.6.7: moving material particles

Another example would be the divergence of the heat flux vector **q**. This time suppose also that there is some generator of heat inside the element (a **source**), generating at a rate of *r* per unit volume, *r* being a scalar field. Again, assuming the element to be small, one takes *r* to be acting at the mid-point of the element, and one considers $r(x_1+\frac{1}{2}\Delta x_1,\cdots)$. Assume a **steady-state** heat flow, so that the (heat) energy within the elemental volume remains constant with time – the law of balance of (heat) energy then requires that the net flow of heat out must equal the heat generated within, so

$$\begin{aligned}\Delta x_1\Delta x_2\Delta x_3\frac{\partial q_1}{\partial x_1}+\Delta x_1\Delta x_2\Delta x_3\frac{\partial q_2}{\partial x_2}+\Delta x_1\Delta x_2\Delta x_3\frac{\partial q_3}{\partial x_3}\\ =\Delta x_1\Delta x_2\Delta x_3\left\{r(x_1,x_2,x_3)+\tfrac{1}{2}\Delta x_1\frac{\partial r}{\partial x_1}+\tfrac{1}{2}\Delta x_2\frac{\partial r}{\partial x_2}+\tfrac{1}{2}\Delta x_3\frac{\partial r}{\partial x_3}\right\}\end{aligned}$$

Dividing through by $\Delta x_1\Delta x_2\Delta x_3$ and taking the limit as $\Delta x_1,\Delta x_2,\Delta x_3\to 0$, one obtains

$$\operatorname{div}\mathbf{q}=r \tag{1.6.19}$$

Here, the divergence of the heat flux vector field can be interpreted as the heat generated (or absorbed) per unit volume per unit time in a temperature field. If the divergence is zero, there is no heat being generated (or absorbed) and the heat leaving the element is equal to the heat entering it.

1.6.7 The Laplacian

Combining Fourier's law of heat conduction (1.6.13), $\mathbf{q}=-k\nabla\theta$, with the energy balance equation (1.6.19), $\operatorname{div}\mathbf{q}=r$, and assuming the conductivity is constant, leads to $-k\,\nabla\cdot\nabla\theta=r$. Now

$$\begin{aligned}\nabla\cdot\nabla\theta&=\mathbf{e}_i\frac{\partial}{\partial x_i}\cdot\left(\frac{\partial\theta}{\partial x_j}\mathbf{e}_j\right)=\frac{\partial}{\partial x_i}\left(\frac{\partial\theta}{\partial x_j}\right)\delta_{ij}=\frac{\partial^2\theta}{\partial x_i^2}\\&=\frac{\partial^2\theta}{\partial x_1^2}+\frac{\partial^2\theta}{\partial x_2^2}+\frac{\partial^2\theta}{\partial x_3^2}\end{aligned} \tag{1.6.20}$$

This expression is called the **Laplacian** of θ. By introducing the Laplacian operator $\nabla^2 \equiv \nabla \cdot \nabla$, one has

$$\nabla^2 \theta = -\frac{r}{k} \tag{1.6.21}$$

This equation governs the steady state heat flow for constant conductivity. In general, the equation $\nabla^2 \phi = a$ is called **Poisson's equation**. When there are no heat sources (or sinks), one has **Laplace's equation**, $\nabla^2 \theta = 0$. Laplace's and Poisson's equation arise in many other mathematical models in mechanics, electromagnetism, etc.

1.6.8 The Curl of a Vector Field

From the definition 1.6.15 and 1.6.14, the **curl** of a vector field $\mathbf{a}(\mathbf{x})$ is the vector field

$$\begin{aligned} \operatorname{curl}\mathbf{a} = \nabla \times \mathbf{a} &= \mathbf{e}_i \frac{\partial}{\partial x_i} \times \left(a_j \mathbf{e}_j\right) \\ &= \frac{\partial a_j}{\partial x_i} \mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk} \frac{\partial a_j}{\partial x_i} \mathbf{e}_k \end{aligned} \qquad \textbf{Curl of a Vector Field} \tag{1.6.22}$$

It can also be expressed in the form

$$\begin{aligned} \operatorname{curl}\mathbf{a} = \nabla \times \mathbf{a} &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ a_1 & a_2 & a_3 \end{vmatrix} \\ &= \varepsilon_{ijk} \frac{\partial a_j}{\partial x_i} \mathbf{e}_k = \varepsilon_{ijk} \frac{\partial a_k}{\partial x_j} \mathbf{e}_i = \varepsilon_{ijk} \frac{\partial a_i}{\partial x_k} \mathbf{e}_j \end{aligned} \tag{1.6.23}$$

Note: the divergence and curl of a vector field are independent of any coordinate system (for example, the divergence of a vector and the length and direction of $\operatorname{curl}\mathbf{a}$ are independent of the coordinate system in use) – these will be re-defined without reference to any particular coordinate system when discussing tensors (see §1.14).

Physical interpretation of the Curl

Consider a particle with position vector **r** and moving with velocity $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, that is, with an angular velocity ω about an axis in the direction of $\boldsymbol{\omega}$. Then {▲Problem 7}

$$\operatorname{curl}\mathbf{v} = \nabla \times (\boldsymbol{\omega} \times \mathbf{r}) = 2\boldsymbol{\omega} \tag{1.6.24}$$

Thus the curl of a vector field is associated with rotational properties. In fact, if **v** is the velocity of a moving fluid, then a small paddle wheel placed in the fluid would tend to rotate in regions where $\operatorname{curl}\mathbf{v} \neq 0$, in which case the velocity field **v** is called a **vortex**

field. The paddle wheel would remain stationary in regions where $\text{curl}\,\mathbf{v} = 0$, in which case the velocity field $\mathbf{v}$ is called **irrotational**.

1.6.9 Identities

Here are some important identities of vector calculus {▲Problem 8}:

$$\begin{aligned} \text{grad}(\phi + \psi) &= \text{grad}\,\phi + \text{grad}\,\psi \\ \text{div}(\mathbf{u} + \mathbf{v}) &= \text{div}\,\mathbf{u} + \text{div}\,\mathbf{v} \\ \text{curl}(\mathbf{u} + \mathbf{v}) &= \text{curl}\,\mathbf{u} + \text{curl}\,\mathbf{v} \end{aligned} \tag{1.6.25}$$

$$\begin{aligned} \text{grad}(\phi\psi) &= \phi\,\text{grad}\,\psi + \psi\,\text{grad}\,\phi \\ \text{div}(\phi\mathbf{u}) &= \phi\,\text{div}\,\mathbf{u} + \text{grad}\,\phi \cdot \mathbf{u} \\ \text{curl}(\phi\mathbf{u}) &= \phi\,\text{curl}\,\mathbf{u} + \text{grad}\,\phi \times \mathbf{u} \\ \text{div}(\mathbf{u} \times \mathbf{v}) &= \mathbf{v} \cdot \text{curl}\,\mathbf{u} - \mathbf{u} \cdot \text{curl}\,\mathbf{v} \\ \text{curl}(\text{grad}\,\phi) &= \mathbf{o} \\ \text{div}(\text{curl}\,\mathbf{u}) &= 0 \\ \text{div}(\lambda\,\text{grad}\,\phi) &= \lambda\nabla^2\phi + \text{grad}\,\lambda \cdot \text{grad}\,\phi \end{aligned} \tag{1.6.26}$$

1.6.10 Cylindrical and Spherical Coordinates

Cartesian coordinates have been used exclusively up to this point. In many practical problems, it is easier to carry out an analysis in terms of cylindrical or spherical coordinates. Differentiation in these coordinate systems is discussed in what follows[10].

Cylindrical Coordinates

Cartesian and cylindrical coordinates are related through (see Fig. 1.6.8)

$$\begin{aligned} x &= r\cos\theta & r &= \sqrt{x^2 + y^2} \\ y &= r\sin\theta\,, & \theta &= \tan^{-1}(y/x) \\ z &= z & z &= z \end{aligned} \tag{1.6.27}$$

Then the Cartesian partial derivatives become

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} \end{aligned} \tag{1.6.28}$$

[10] this section also serves as an introduction to the more general topic of **Curvilinear Coordinates** covered in §1.16-§1.19

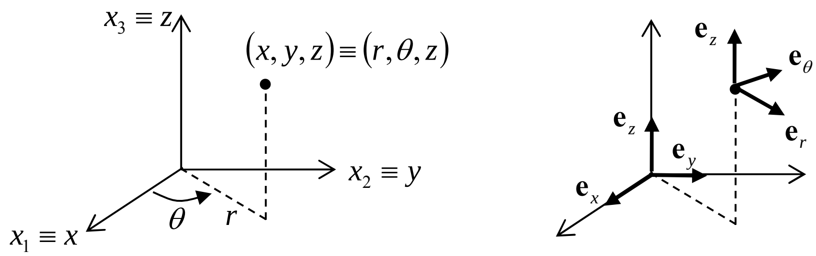


Figure 1.6.8: cylindrical coordinates

The base vectors are related through

$$\begin{aligned} \mathbf{e}_x &= \mathbf{e}_r \cos\theta - \mathbf{e}_\theta \sin\theta & \mathbf{e}_r &= \mathbf{e}_x \cos\theta + \mathbf{e}_y \sin\theta \\ \mathbf{e}_y &= \mathbf{e}_r \sin\theta + \mathbf{e}_\theta \cos\theta, & \mathbf{e}_\theta &= -\mathbf{e}_x \sin\theta + \mathbf{e}_y \cos\theta \\ \mathbf{e}_z &= \mathbf{e}_z & \mathbf{e}_z &= \mathbf{e}_z \end{aligned} \tag{1.6.29}$$

so that from Eqn. 1.6.14, after some algebra, the Nabla operator in cylindrical coordinates reads as {▲Problem 9}

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z} \tag{1.6.30}$$

which allows one to take the gradient of a scalar field in cylindrical coordinates:

$$\nabla\phi = \frac{\partial \phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\mathbf{e}_\theta + \frac{\partial \phi}{\partial z}\mathbf{e}_z \tag{1.6.31}$$

Cartesian base vectors are independent of position. However, the cylindrical base vectors, although they are always of unit magnitude, change direction with position. In particular, the directions of the base vectors $\mathbf{e}_r, \mathbf{e}_\theta$ depend on θ, and so these base vectors have derivatives with respect to θ: from Eqn. 1.6.29,

$$\begin{aligned} \frac{\partial}{\partial \theta}\mathbf{e}_r &= \mathbf{e}_\theta \\ \frac{\partial}{\partial \theta}\mathbf{e}_\theta &= -\mathbf{e}_r \end{aligned} \tag{1.6.32}$$

with all other derivatives of the base vectors with respect to r, θ, z equal to zero.

The divergence can now be evaluated:

$$\nabla \cdot \mathbf{v} = \left(\mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z} \right) \cdot \left(v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_z \mathbf{e}_z \right)$$
$$= \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \qquad (1.6.33)$$

Similarly the curl of a vector and the Laplacian of a scalar are {▲Problem 10}

$$\nabla \times \mathbf{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} \right)\mathbf{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \right)\mathbf{e}_\theta + \left[\frac{1}{r}\left(\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta} \right) \right]\mathbf{e}_z$$
$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} \qquad (1.6.34)$$

Spherical Coordinates

Cartesian and spherical coordinates are related through (see Fig. 1.6.9)

$$\begin{aligned} x &= r\sin\theta\cos\phi & \quad r &= \sqrt{x^2+y^2+z^2} \\ y &= r\sin\theta\sin\phi\,, & \quad \theta &= \tan^{-1}\left(\sqrt{x^2+y^2}\,/\,z\right) \\ z &= r\cos\theta & \quad \phi &= \tan^{-1}(y/x) \end{aligned} \qquad (1.6.35)$$

and the base vectors are related through

$$\begin{aligned} \mathbf{e}_x &= \mathbf{e}_r \sin\theta\cos\phi + \mathbf{e}_\theta \cos\theta\cos\phi - \mathbf{e}_\phi \sin\phi \\ \mathbf{e}_y &= \mathbf{e}_r \sin\theta\sin\phi + \mathbf{e}_\theta \cos\theta\sin\phi + \mathbf{e}_\phi \cos\phi \\ \mathbf{e}_z &= \mathbf{e}_r \cos\theta - \mathbf{e}_\theta \sin\theta \\ \\ \mathbf{e}_r &= \mathbf{e}_x \sin\theta\cos\phi + \mathbf{e}_y \sin\theta\sin\phi + \mathbf{e}_z \cos\theta \\ \mathbf{e}_\theta &= \mathbf{e}_x \cos\theta\cos\phi + \mathbf{e}_y \cos\theta\sin\phi - \mathbf{e}_z \sin\theta \\ \mathbf{e}_\phi &= -\mathbf{e}_x \sin\phi + \mathbf{e}_y \cos\phi \end{aligned} \qquad (1.6.36)$$

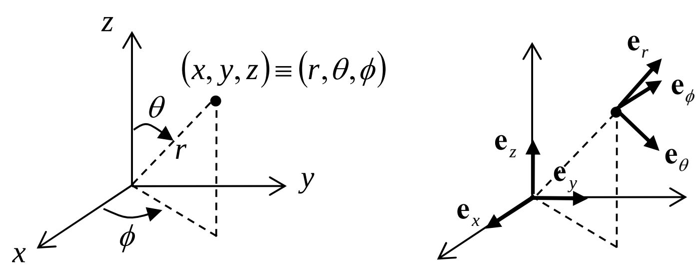


Figure 1.6.9: spherical coordinates

In this case the non-zero derivatives of the base vectors are

$$\begin{aligned}\frac{\partial}{\partial\theta}\mathbf{e}_r &= \mathbf{e}_\theta \\ \frac{\partial}{\partial\theta}\mathbf{e}_\theta &= -\mathbf{e}_r\end{aligned}, \quad \begin{aligned}\frac{\partial}{\partial\phi}\mathbf{e}_r &= \sin\theta\mathbf{e}_\phi \\ \frac{\partial}{\partial\phi}\mathbf{e}_\theta &= \cos\theta\mathbf{e}_\phi \\ \frac{\partial}{\partial\phi}\mathbf{e}_\phi &= -\sin\theta\mathbf{e}_r - \cos\theta\mathbf{e}_\theta\end{aligned} \tag{1.6.37}$$

and it can then be shown that {▲Problem 11}

$$\begin{aligned}\nabla\varphi &= \frac{\partial\varphi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\phi}\mathbf{e}_\phi \\ \nabla\cdot\mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} \\ \nabla^2\varphi &= \frac{\partial^2\varphi}{\partial r^2} + \frac{2}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\varphi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2}\end{aligned} \tag{1.6.38}$$

1.6.11 The Directional Derivative

Consider a function $\phi(\mathbf{x})$. The directional derivative of φ in the direction of some vector $\mathbf{w}$ is the change in φ in that direction. Now the difference between its values at position $\mathbf{x}$ and $\mathbf{x}+\mathbf{w}$ is, Fig. 1.6.10,

$$d\phi = \phi(\mathbf{x}+\mathbf{w}) - \phi(\mathbf{x}) \tag{1.6.39}$$

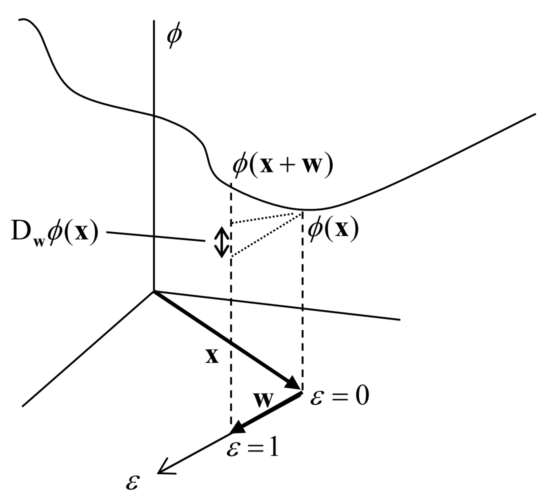


Figure 1.6.10: the directional derivative

An approximation to $d\phi$ can be obtained by introducing a parameter ε and by considering the function $\phi(\mathbf{x}+\varepsilon\mathbf{w})$; one has $\phi(\mathbf{x}+\varepsilon\mathbf{w})_{\varepsilon=0} = \phi(\mathbf{x})$ and $\phi(\mathbf{x}+\varepsilon\mathbf{w})_{\varepsilon=1} = \phi(\mathbf{x}+\mathbf{w})$.

If one treats ϕ as a function of ε, a Taylor's series about $\varepsilon = 0$ gives

$$\phi(\varepsilon) = \phi(0) + \varepsilon \left.\frac{d\phi(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} + \frac{\varepsilon^2}{2}\left.\frac{d^2\phi(\varepsilon)}{d\varepsilon^2}\right|_{\varepsilon=0} + \cdots$$

or, writing it as a function of $\mathbf{x}+\varepsilon\mathbf{w}$,

$$\phi(\mathbf{x}+\varepsilon\mathbf{w}) = \phi(\mathbf{x}) + \varepsilon \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \phi(\mathbf{x}+\varepsilon\mathbf{w}) + \cdots$$

By setting $\varepsilon = 1$, the derivative here can be seen to be a linear approximation to the increment $d\phi$, Eqn. 1.6.39. This is defined as the **directional derivative** of the function $\phi(\mathbf{x})$ at the point **x** in the direction of **w**, and is denoted by

$$\boxed{\partial_{\mathbf{x}}\phi[\mathbf{w}] = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \phi(\mathbf{x}+\varepsilon\mathbf{w})} \qquad \textbf{The Directional Derivative} \qquad (1.6.40)$$

The directional derivative is also written as $\mathrm{D}_{\mathbf{w}}\phi(\mathbf{x})$.

The power of the directional derivative as defined by Eqn. 1.6.40 is its generality, as seen in the following example.

Example (the Directional Derivative of the Determinant)

Consider the directional derivative of the determinant of the 2×2 matrix **A**, in the direction of a second matrix **T** (the word "direction" is obviously used loosely in this context). One has

$$\begin{aligned}
\partial_{\mathbf{A}}(\det \mathbf{A})[\mathbf{T}] &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \det(\mathbf{A}+\varepsilon\mathbf{T}) \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \left[(A_{11}+\varepsilon T_{11})(A_{22}+\varepsilon T_{22}) - (A_{12}+\varepsilon T_{12})(A_{21}+\varepsilon T_{21})\right] \\
&= A_{11}T_{22} + A_{22}T_{11} - A_{12}T_{21} - A_{21}T_{12}
\end{aligned}$$

■

The Directional Derivative and The Gradient

Consider a scalar-valued function ϕ of a vector **z**. Let **z** be a function of a parameter ε, $\phi \equiv \phi(z_1(\varepsilon), z_2(\varepsilon), z_3(\varepsilon))$. Then

$$\frac{d\phi}{d\varepsilon} = \frac{\partial \phi}{\partial z_i}\frac{dz_i}{d\varepsilon} = \frac{\partial \phi}{\partial \mathbf{z}} \cdot \frac{d\mathbf{z}}{d\varepsilon}$$

Thus, with $\mathbf{z} = \mathbf{x} + \varepsilon \mathbf{w}$,

$$\partial_{\mathbf{x}} \phi[\mathbf{w}] = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \phi(\mathbf{z}(\varepsilon)) = \left(\frac{\partial \phi}{\partial \mathbf{z}} \cdot \frac{d\mathbf{z}}{d\varepsilon}\right)_{\varepsilon=0} = \frac{\partial \phi}{\partial \mathbf{x}} \cdot \mathbf{w} \tag{1.6.41}$$

which can be compared with Eqn. 1.6.11. Note that for Eqns. 1.6.11 and 1.6.41 to be consistent definitions of the directional derivative, **w** here should be a *unit* vector.

1.6.12 Formal Treatment of Vector Calculus

The calculus of vectors is now treated more formally in what follows, following on from the introductory section in §1.2. Consider a vector **h**, an element of the Euclidean vector space *E*, $\mathbf{h} \in E$. In order to be able to speak of limits as elements become "small" or "close" to each other in this space, one requires a norm. Here, take the standard Euclidean norm on *E*, Eqn. 1.2.8,

$$\|\mathbf{h}\| \equiv \sqrt{\langle \mathbf{h}, \mathbf{h} \rangle} = \sqrt{\mathbf{h} \cdot \mathbf{h}} \tag{1.6.42}$$

Consider next a scalar function $f : E \to R$. If there is a constant $M > 0$ such that $|f(\mathbf{h})| \le M\|\mathbf{h}\|$ as $\mathbf{h} \to \mathbf{o}$, then one writes

$$f(\mathbf{h}) = O(\|\mathbf{h}\|) \quad \text{as} \quad \mathbf{h} \to \mathbf{o} \tag{1.6.43}$$

This is called the **Big Oh** (or **Landau**) notation. Eqn. 1.6.43 states that $|f(\mathbf{h})|$ goes to zero at least as fast as $\|\mathbf{h}\|$. An expression such as

$$f(\mathbf{h}) = g(\mathbf{h}) + O(\|\mathbf{h}\|) \tag{1.6.44}$$

then means that $|f(\mathbf{h}) - g(\mathbf{h})|$ is smaller than $\|\mathbf{h}\|$ for $\mathbf{h}$ sufficiently close to **o**.

Similarly, if

$$\frac{f(\mathbf{h})}{\|\mathbf{h}\|} \to 0 \quad \text{as} \quad \mathbf{h} \to \mathbf{o} \tag{1.6.45}$$

then one writes $f(\mathbf{h}) = o(\|\mathbf{h}\|)$ as $\mathbf{h} \to \mathbf{o}$. This implies that $|f(\mathbf{h})|$ goes to zero faster than $\|\mathbf{h}\|$.

A **field** is a function which is defined in a Euclidean (point) space E^3. A **scalar field** is then a function $f : E^3 \to R$. A scalar field is **differentiable** at a point $\mathbf{x} \in E^3$ if there exists a vector $Df(\mathbf{x}) \in E$ such that

$$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + Df(\mathbf{x}) \cdot \mathbf{h} + o(\|\mathbf{h}\|) \quad \text{for all} \quad \mathbf{h} \in E \tag{1.6.46}$$

In that case, the vector $Df(\mathbf{x})$ is called the **derivative** (or **gradient**) of f at $\mathbf{x}$ (and is given the symbol $\nabla f(\mathbf{x})$).

Now setting $\mathbf{h} = \varepsilon \mathbf{w}$ in 1.6.46, where $\mathbf{w} \in E$ is a unit vector, dividing through by ε and taking the limit as $\varepsilon \to 0$, one has the equivalent statement

$$\nabla f(\mathbf{x}) \cdot \mathbf{w} = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} f(\mathbf{x}+\varepsilon \mathbf{w}) \quad \text{for all} \quad \mathbf{w} \in E \tag{1.6.47}$$

which is 1.6.41. In other words, for the derivative to exist, the scalar field must have a directional derivative in all directions at $\mathbf{x}$.

Using the chain rule as in §1.6.11, Eqn. 1.6.47 can be expressed in terms of the Cartesian basis $\{\mathbf{e}_i\}$,

$$\nabla f(\mathbf{x}) \cdot \mathbf{w} = \frac{\partial f}{\partial x_i} w_i = \frac{\partial f}{\partial x_i} \mathbf{e}_i \cdot w_j \mathbf{e}_j \tag{1.6.48}$$

This must be true for all $\mathbf{w}$ and so, in a Cartesian basis,

$$\nabla f(\mathbf{x}) = \frac{\partial f}{\partial x_i} \mathbf{e}_i \tag{1.6.49}$$

which is Eqn. 1.6.9.

1.6.13 Problems

1. A particle moves along a curve in space defined by
$$\mathbf{r} = (t^3 - 4t)\mathbf{e}_1 + (t^2 + 4t)\mathbf{e}_2 + (8t^2 - 3t^3)\mathbf{e}_3$$
Here, t is time. Find
(i) a unit tangent vector at $t = 2$
(ii) the magnitudes of the tangential and normal components of acceleration at $t = 2$
2. Use the index notation (1.3.12) to show that $\frac{d}{dt}(\mathbf{v} \times \mathbf{a}) = \mathbf{v} \times \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{v}}{dt} \times \mathbf{a}$. Verify this result for $\mathbf{v} = 3t\mathbf{e}_1 - t^2\mathbf{e}_3$, $\mathbf{a} = t^2\mathbf{e}_1 + t\mathbf{e}_2$. [Note: the permutation symbol and the unit vectors are independent of t; the components of the vectors are scalar functions of t which can be differentiated in the usual way, for example by using the product rule of differentiation.]

3. The density distribution throughout a material is given by $\rho = 1 + \mathbf{x} \cdot \mathbf{x}$.
   (i) what sort of function is this?
   (ii) the density is given in symbolic notation - write it in index notation
   (iii) evaluate the gradient of ρ
   (iv) give a unit vector in the direction in which the density is increasing the most
   (v) give a unit vector in *any* direction in which the density is not increasing
   (vi) take any unit vector other than the base vectors and the other vectors you used above and calculate $d\rho / dx$ in the direction of this unit vector
   (vii) evaluate and sketch all these quantities for the point (2,1).
   In parts (iii-iv), give your answer in (a) symbolic, (b) index, and (c) full notation.
4. Consider the scalar field defined by $\phi = x^2 + 3yx + 2z$.
   (i) find the unit normal to the surface of constant ϕ at the origin (0,0,0)
   (ii) what is the maximum value of the directional derivative of ϕ at the origin?
   (iii) evaluate $d\phi / dx$ at the origin if $d\mathbf{x} = ds(\mathbf{e}_1 + \mathbf{e}_3)$.
5. If $\mathbf{u} = x_1 x_2 x_3 \mathbf{e}_1 + x_1 x_2 \mathbf{e}_2 + x_1 \mathbf{e}_3$, determine $\operatorname{div} \mathbf{u}$ and $\operatorname{curl} \mathbf{u}$.
6. Determine the constant *a* so that the vector
$$\mathbf{v} = (x_1 + 3x_2)\mathbf{e}_1 + (x_2 - 2x_3)\mathbf{e}_2 + (x_1 + ax_3)\mathbf{e}_3$$
   is solenoidal.
7. Show that $\operatorname{curl} \mathbf{v} = 2\boldsymbol{\omega}$ (see also Problem 9 in §1.1).
8. Verify the identities (1.6.25-26).
9. Use (1.6.14) to derive the Nabla operator in cylindrical coordinates (1.6.30).
10. Derive Eqn. (1.6.34), the curl of a vector and the Laplacian of a scalar in the cylindrical coordinates.
11. Derive (1.6.38), the gradient, divergence and Laplacian in spherical coordinates.
12. Show that the directional derivative $\mathrm{D}_{\mathbf{v}}\phi(\mathbf{u})$ of the scalar-valued function of a vector $\phi(\mathbf{u}) = \mathbf{u} \cdot \mathbf{u}$, in the direction **v**, is $2\mathbf{u} \cdot \mathbf{v}$.
13. Show that the directional derivative of the functional
$$U(v(x)) = \frac{1}{2}\int_0^l EI\left(\frac{d^2 v}{dx^2}\right)^2 dx - \int_0^l p(x)v(x)dx$$
   in the direction of $\omega(x)$ is given by
$$\int_0^l EI \frac{d^2 v(x)}{dx^2}\frac{d^2 \omega(x)}{dx^2} dx - \int_0^l p(x)\omega(x)dx .$$

16

Vector Calculus

16.1 Vector Fields

This chapter is concerned with applying calculus in the context of **vector fields**. A two-dimensional vector field is a function f that maps each point (x, y) in $\mathbb{R}^2$ to a two-dimensional vector $\langle u, v\rangle$, and similarly a three-dimensional vector field maps (x, y, z) to $\langle u, v, w\rangle$. Since a vector has no position, we typically indicate a vector field in graphical form by placing the vector $f(x, y)$ with its tail at (x, y). Figure 16.1.1 shows a representation of the vector field $f(x, y) = \langle -x/\sqrt{x^2 + y^2 + 4}, y/\sqrt{x^2 + y^2 + 4}\rangle$. For such a graph to be readable, the vectors must be fairly short, which is accomplished by using a different scale for the vectors than for the axes. Such graphs are thus useful for understanding the sizes of the vectors relative to each other but not their absolute size.

Vector fields have many important applications, as they can be used to represent many physical quantities: the vector at a point may represent the strength of some force (gravity, electricity, magnetism) or a velocity (wind speed or the velocity of some other fluid).

We have already seen a particularly important kind of vector field—the gradient. Given a function $f(x, y)$, recall that the gradient is $\langle f_x(x, y), f_y(x, y)\rangle$, a vector that depends on (is a function of) x and y. We usually picture the gradient vector with its tail at (x, y), pointing in the direction of maximum increase. Vector fields that are gradients have some particularly nice properties, as we will see. An important example is

$$\mathbf{F} = \left\langle \frac{-x}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-y}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-z}{(x^2 + y^2 + z^2)^{3/2}} \right\rangle,$$

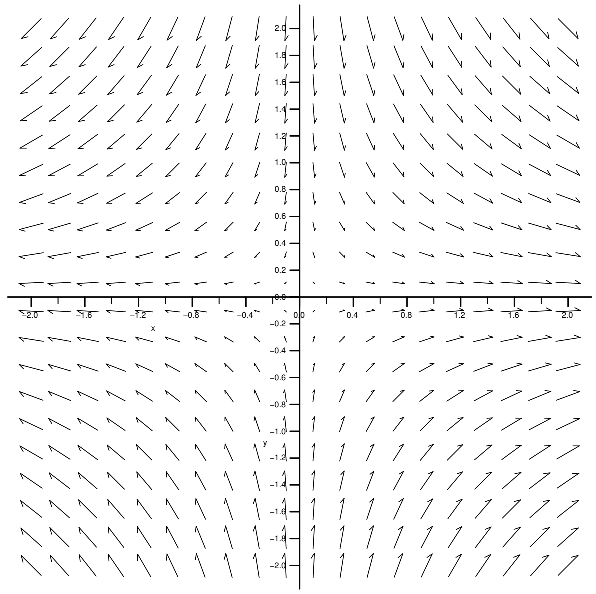


Figure 16.1.1 A vector field.

which points from the point (x, y, z) toward the origin and has length

$$\frac{\sqrt{x^2+y^2+z^2}}{(x^2+y^2+z^2)^{3/2}} = \frac{1}{(\sqrt{x^2+y^2+z^2})^2},$$

which is the reciprocal of the square of the distance from (x, y, z) to the origin—in other words, $\mathbf{F}$ is an "inverse square law". The vector $\mathbf{F}$ is a gradient:

$$\mathbf{F} = \nabla \frac{1}{\sqrt{x^2+y^2+z^2}}, \tag{16.1.1}$$

which turns out to be extremely useful.

Exercises 16.1.

Sketch the vector fields; check your work with Sage's `plot_vector_field` function.

1. $\langle x, y\rangle$

2. $\langle -x, -y\rangle$

3. $\langle x, -y\rangle$

4. $\langle \sin x, \cos y\rangle$

5. $\langle y, 1/x\rangle$

6. $\langle x+1, x+3\rangle$

7. Verify equation 16.1.1.

16.2 Line Integrals

We have so far integrated "over" intervals, areas, and volumes with single, double, and triple integrals. We now investigate integration over or "along" a curve—"line integrals" are really "curve integrals".

As with other integrals, a geometric example may be easiest to understand. Consider the function $f = x + y$ and the parabola $y = x^2$ in the x-y plane, for $0 \le x \le 2$. Imagine that we extend the parabola up to the surface f, to form a curved wall or curtain, as in figure 16.2.1. What is the area of the surface thus formed? We already know one way to compute surface area, but here we take a different approach that is more useful for the problems to come.

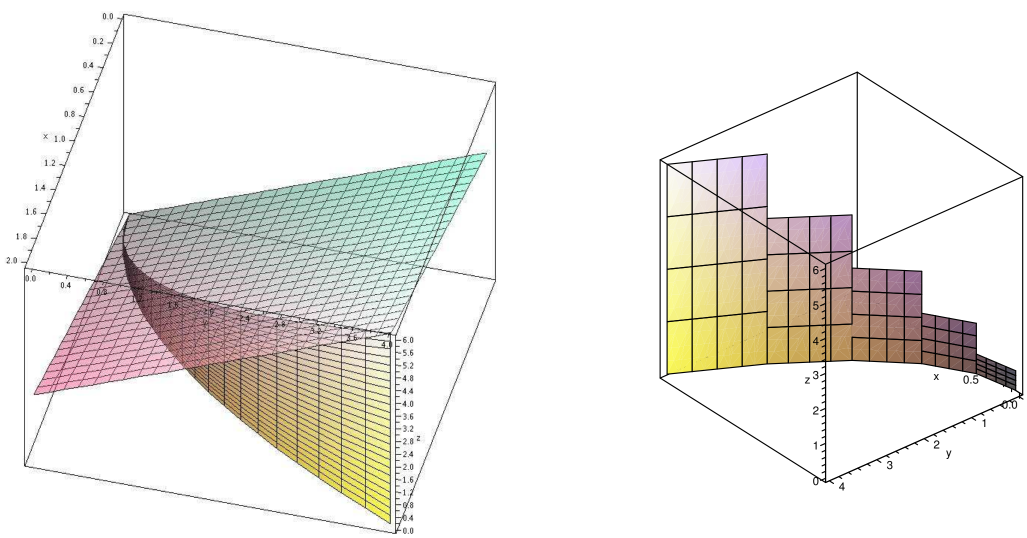


Figure 16.2.1 Approximating the area under a curve. (AP)

As usual, we start by thinking about how to approximate the area. We pick some points along the part of the parabola we're interested in, and connect adjacent points by straight lines; when the points are close together, the length of each line segment will be close to the length along the parabola. Using each line segment as the base of a rectangle, we choose the height to be the height of the surface f above the line segment. If we add up the areas of these rectangles, we get an approximation to the desired area, and in the limit this sum turns into an integral.

Typically the curve is in vector form, or can easily be put in vector form; in this example we have $\mathbf{v}(t) = \langle t, t^2 \rangle$. Then as we have seen in section 13.3 on arc length, the length of one of the straight line segments in the approximation is approximately

$ds = |\mathbf{v}'|\,dt = \sqrt{1+4t^2}\,dt$, so the integral is

$$\int_0^2 f(t,t^2)\sqrt{1+4t^2}\,dt = \int_0^2 (t+t^2)\sqrt{1+4t^2}\,dt = \frac{167}{48}\sqrt{17} - \frac{1}{12} - \frac{1}{64}\ln(4+\sqrt{17}).$$

This integral of a function along a curve C is often written in abbreviated form as

$$\int_C f(x,y)\,ds.$$

EXAMPLE 16.2.1 Compute $\int_C ye^x\,ds$ where C is the line segment from $(1,2)$ to $(4,7)$.

We write the line segment as a vector function: $\mathbf{v} = \langle 1,2\rangle + t\langle 3,5\rangle$, $0 \le t \le 1$, or in parametric form $x = 1+3t$, $y = 2+5t$. Then

$$\int_C ye^x\,ds = \int_0^1 (2+5t)e^{1+3t}\sqrt{3^2+5^2}\,dt = \frac{16}{9}\sqrt{34}e^4 - \frac{1}{9}\sqrt{34}\,e.$$

□

All of these ideas extend to three dimensions in the obvious way.

EXAMPLE 16.2.2 Compute $\int_C x^2z\,ds$ where C is the line segment from $(0,6,-1)$ to $(4,1,5)$.

We write the line segment as a vector function: $\mathbf{v} = \langle 0,6,-1\rangle + t\langle 4,-5,6\rangle$, $0 \le t \le 1$, or in parametric form $x = 4t$, $y = 6-5t$, $z = -1+6t$. Then

$$\int_C x^2z\,ds = \int_0^1 (4t)^2(-1+6t)\sqrt{16+25+36}\,dt = 16\sqrt{77}\int_0^1 -t^2+6t^3\,dt = \frac{56}{3}\sqrt{77}.$$

□

Now we turn to a perhaps more interesting example. Recall that in the simplest case, the work done by a force on an object is equal to the magnitude of the force times the distance the object moves; this assumes that the force is constant and in the direction of motion. We have already dealt with examples in which the force is not constant; now we are prepared to examine what happens when the force is not parallel to the direction of motion.

We have already examined the idea of components of force, in example 12.3.4: the component of a force $\mathbf{F}$ in the direction of a vector $\mathbf{v}$ is

$$\frac{\mathbf{F}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v},$$

the projection of $\mathbf{F}$ onto $\mathbf{v}$. The length of this vector, that is, the magnitude of the force in the direction of $\mathbf{v}$, is

$$\frac{\mathbf{F}\cdot\mathbf{v}}{|\mathbf{v}|},$$

the scalar projection of $\mathbf{F}$ onto $\mathbf{v}$. If an object moves subject to this (constant) force, in the direction of $\mathbf{v}$, over a distance equal to the length of $\mathbf{v}$, the work done is

$$\frac{\mathbf{F}\cdot\mathbf{v}}{|\mathbf{v}|}|\mathbf{v}| = \mathbf{F}\cdot\mathbf{v}.$$

Thus, work in the vector setting is still "force times distance", except that "times" means "dot product".

If the force varies from point to point, it is represented by a vector field $\mathbf{F}$; the displacement vector $\mathbf{v}$ may also change, as an object may follow a curving path in two or three dimensions. Suppose that the path of an object is given by a vector function $\mathbf{r}(t)$; at any point along the path, the (small) tangent vector $\mathbf{r}'\,\Delta t$ gives an approximation to its motion over a short time Δt, so the work done during that time is approximately $\mathbf{F}\cdot\mathbf{r}'\,\Delta t$; the total work over some time period is then

$$\int_{t_0}^{t_1} \mathbf{F}\cdot\mathbf{r}'\,dt.$$

It is useful to rewrite this in various ways at different times. We start with

$$\int_{t_0}^{t_1} \mathbf{F}\cdot\mathbf{r}'\,dt = \int_C \mathbf{F}\cdot\,d\mathbf{r},$$

abbreviating $\mathbf{r}'\,dt$ by $d\mathbf{r}$. Or we can write

$$\int_{t_0}^{t_1} \mathbf{F}\cdot\mathbf{r}'\,dt = \int_{t_0}^{t_1} \mathbf{F}\cdot\frac{\mathbf{r}'}{|\mathbf{r}'|}|\mathbf{r}'|\,dt = \int_{t_0}^{t_1} \mathbf{F}\cdot\mathbf{T}\,|\mathbf{r}'|\,dt = \int_C \mathbf{F}\cdot\mathbf{T}\,ds,$$

using the unit tangent vector $\mathbf{T}$, abbreviating $|\mathbf{r}'|\,dt$ as ds, and indicating the path of the object by C. In other words, work is computed using a particular line integral of the form we have considered. Alternately, we sometimes write

$$\begin{aligned}\int_C \mathbf{F}\cdot\mathbf{r}'\,dt &= \int_C \langle f,g,h\rangle\cdot\langle x',y',z'\rangle\,dt = \int_C \left(f\frac{dx}{dt}+g\frac{dy}{dt}+h\frac{dz}{dt}\right)\,dt\\ &= \int_C f\,dx+g\,dy+h\,dz = \int_C f\,dx+\int_C g\,dy+\int_C h\,dz,\end{aligned}$$

and similarly for two dimensions, leaving out references to z.

EXAMPLE 16.2.3 Suppose an object moves from $(-1,1)$ to $(2,4)$ along the path $\mathbf{r}(t)=\langle t,t^2\rangle$, subject to the force $\mathbf{F}=\langle x\sin y, y\rangle$. Find the work done.

We can write the force in terms of t as $\langle t\sin(t^2),t^2\rangle$, and compute $\mathbf{r}'(t)=\langle 1,2t\rangle$, and then the work is

$$\int_{-1}^{2}\langle t\sin(t^2),t^2\rangle\cdot\langle 1,2t\rangle\,dt = \int_{-1}^{2} t\sin(t^2)+2t^3\,dt = \frac{15}{2}+\frac{\cos(1)-\cos(4)}{2}.$$

Alternately, we might write

$$\int_C x\sin y\,dx + \int_C y\,dy = \int_{-1}^{2} x\sin(x^2)\,dx + \int_1^4 y\,dy = -\frac{\cos(4)}{2}+\frac{\cos(1)}{2}+\frac{16}{2}-\frac{1}{2}$$

getting the same answer. □

Exercises 16.2.

1. Compute $\int_C xy^2\,ds$ along the line segment from $(1,2,0)$ to $(2,1,3)$. ⇒

2. Compute $\int_C \sin x\,ds$ along the line segment from $(-1,2,1)$ to $(1,2,5)$. ⇒

3. Compute $\int_C z\cos(xy)\,ds$ along the line segment from $(1,0,1)$ to $(2,2,3)$. ⇒

4. Compute $\int_C \sin x\,dx+\cos y\,dy$ along the top half of the unit circle, from $(1,0)$ to $(-1,0)$. ⇒

5. Compute $\int_C xe^y\,dx + x^2y\,dy$ along the line segment $y=3$, $0\le x\le 2$. ⇒

6. Compute $\int_C xe^y\,dx + x^2y\,dy$ along the line segment $x=4$, $0\le y\le 4$. ⇒

7. Compute $\int_C xe^y\,dx + x^2y\,dy$ along the curve $x=3t$, $y=t^2$, $0\le t\le 1$. ⇒

8. Compute $\int_C xe^y\,dx + x^2y\,dy$ along the curve $\langle e^t,e^t\rangle$, $-1\le t\le 1$. ⇒

9. Compute $\int_C \langle\cos x,\sin y\rangle\cdot d\mathbf{r}$ along the curve $\langle t,t\rangle$, $0\le t\le 1$. ⇒

10. Compute $\int_C \langle 1/xy, 1/(x+y)\rangle \cdot d\mathbf{r}$ along the path from $(1,1)$ to $(3,1)$ to $(3,6)$ using straight line segments. ⇒

11. Compute $\int_C \langle 1/xy, 1/(x+y)\rangle \cdot d\mathbf{r}$ along the curve $\langle 2t, 5t\rangle$, $1 \le t \le 4$. ⇒

12. Compute $\int_C \langle 1/xy, 1/(x+y)\rangle \cdot d\mathbf{r}$ along the curve $\langle t, t^2\rangle$, $1 \le t \le 4$. ⇒

13. Compute $\int_C yz\,dx + xz\,dy + xy\,dz$ along the curve $\langle t, t^2, t^3\rangle$, $0 \le t \le 1$. ⇒

14. Compute $\int_C yz\,dx + xz\,dy + xy\,dz$ along the curve $\langle \cos t, \sin t, \tan t\rangle$, $0 \le t \le \pi$. ⇒

15. An object moves from $(1,1)$ to $(4,8)$ along the path $\mathbf{r}(t) = \langle t^2, t^3\rangle$, subject to the force $\mathbf{F} = \langle x^2, \sin y\rangle$. Find the work done. ⇒

16. An object moves along the line segment from $(1,1)$ to $(2,5)$, subject to the force $\mathbf{F} = \langle x/(x^2+y^2), y/(x^2+y^2)\rangle$. Find the work done. ⇒

17. An object moves along the parabola $\mathbf{r}(t) = \langle t, t^2\rangle$, $0 \le t \le 1$, subject to the force $\mathbf{F} = \langle 1/(y+1), -1/(x+1)\rangle$. Find the work done. ⇒

18. An object moves along the line segment from $(0,0,0)$ to $(3,6,10)$, subject to the force $\mathbf{F} = \langle x^2, y^2, z^2\rangle$. Find the work done. ⇒

19. An object moves along the curve $\mathbf{r}(t) = \langle \sqrt{t}, 1/\sqrt{t}, t\rangle$ $1 \le t \le 4$, subject to the force $\mathbf{F} = \langle y, z, x\rangle$. Find the work done. ⇒

20. An object moves from $(1,1,1)$ to $(2,4,8)$ along the path $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$, subject to the force $\mathbf{F} = \langle \sin x, \sin y, \sin z\rangle$. Find the work done. ⇒

21. An object moves from $(1,0,0)$ to $(-1,0,\pi)$ along the path $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$, subject to the force $\mathbf{F} = \langle y^2, y^2, xz\rangle$. Find the work done. ⇒

22. Give an example of a non-trivial force field $\mathbf{F}$ and non-trivial path $\mathbf{r}(t)$ for which the total work done moving along the path is zero.

16.3 The Fundamental Theorem of Line Integrals

One way to write the Fundamental Theorem of Calculus (7.2.1) is:

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

That is, to compute the integral of a derivative f' we need only compute the values of f at the endpoints. Something similar is true for line integrals of a certain form.

THEOREM 16.3.1 Fundamental Theorem of Line Integrals Suppose a curve C is given by the vector function $\mathbf{r}(t)$, with $\mathbf{a} = \mathbf{r}(a)$ and $\mathbf{b} = \mathbf{r}(b)$. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a}),$$

provided that $\mathbf{r}$ is sufficiently nice.

Proof. We write $\mathbf{r} = \langle x(t), y(t), z(t)\rangle$, so that $\mathbf{r}' = \langle x'(t), y'(t), z'(t)\rangle$. Also, we know that $\nabla f = \langle f_x, f_y, f_z\rangle$. Then

$$\int_C \nabla f \cdot d\mathbf{r} = \int_a^b \langle f_x, f_y, f_z\rangle \cdot \langle x'(t), y'(t), z'(t)\rangle\, dt = \int_a^b f_x x' + f_y y' + f_z z'\, dt.$$

By the chain rule (see section 14.4) $f_x x' + f_y y' + f_z z' = df/dt$, where f in this context means $f(x(t), y(t), z(t))$, a function of t. In other words, all we have is

$$\int_a^b f'(t)\, dt = f(b) - f(a).$$

In this context, $f(a) = f(x(a), y(a), z(a))$. Since $\mathbf{a} = \mathbf{r}(a) = \langle x(a), y(a), z(a)\rangle$, we can write $f(a) = f(\mathbf{a})$—this is a bit of a cheat, since we are simultaneously using f to mean $f(t)$ and $f(x, y, z)$, and since $f(x(a), y(a), z(a))$ is not technically the same as $f(\langle x(a), y(a), z(a)\rangle)$, but the concepts are clear and the different uses are compatible. Doing the same for b, we get

$$\int_C \nabla f \cdot d\mathbf{r} = \int_a^b f'(t)\, dt = f(b) - f(a) = f(\mathbf{b}) - f(\mathbf{a}).$$ ■

This theorem, like the Fundamental Theorem of Calculus, says roughly that if we integrate a "derivative-like function" (f' or ∇f) the result depends only on the values of the original function (f) at the endpoints.

If a vector field $\mathbf{F}$ is the gradient of a function, $\mathbf{F} = \nabla f$, we say that $\mathbf{F}$ is a **conservative vector field**. If $\mathbf{F}$ is a conservative force field, then the integral for work, $\int_C \mathbf{F} \cdot d\mathbf{r}$, is in the form required by the Fundamental Theorem of Line Integrals. This means that in a conservative force field, the amount of work required to move an object from point $\mathbf{a}$ to point $\mathbf{b}$ depends only on those points, not on the path taken between them.

EXAMPLE 16.3.2 An object moves in the force field

$$\mathbf{F} = \left\langle \frac{-x}{(x^2+y^2+z^2)^{3/2}}, \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \frac{-z}{(x^2+y^2+z^2)^{3/2}} \right\rangle,$$

along the curve $\mathbf{r} = \langle 1+t, t^3, t\cos(\pi t)\rangle$ as t ranges from 0 to 1. Find the work done by the force on the object.

The straightforward way to do this involves substituting the components of $\mathbf{r}$ into $\mathbf{F}$, forming the dot product $\mathbf{F} \cdot \mathbf{r}'$, and then trying to compute the integral, but this integral is extraordinarily messy, perhaps impossible to compute. But since $\mathbf{F} = \nabla(1/\sqrt{x^2+y^2+z^2})$ we need only substitute:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \left.\frac{1}{\sqrt{x^2+y^2+z^2}}\right|_{(1,0,0)}^{(2,1,-1)} = \frac{1}{\sqrt{6}} - 1.$$

□

Another immediate consequence of the Fundamental Theorem involves **closed paths**. A path C is closed if it forms a loop, so that traveling over the C curve brings you back to the starting point. If C is a closed path, we can integrate around it starting at any point $\mathbf{a}$; since the starting and ending points are the same,

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{a}) - f(\mathbf{a}) = 0.$$

For example, in a gravitational field (an inverse square law field) the amount of work required to move an object around a closed path is zero. Of course, it's only the *net* amount of work that is zero. It may well take a great deal of work to get from point $\mathbf{a}$ to point $\mathbf{b}$, but then the return trip will "produce" work. For example, it takes work to pump water from a lower to a higher elevation, but if you then let gravity pull the water back down, you can recover work by running a water wheel or generator. (In the real world you won't recover all the work because of various losses along the way.)

To make use of the Fundamental Theorem of Line Integrals, we need to be able to spot conservative vector fields $\mathbf{F}$ and to compute f so that $\mathbf{F} = \nabla f$. Suppose that $\mathbf{F} = \langle P, Q\rangle = \nabla f$. Then $P = f_x$ and $Q = f_y$, and provided that f is sufficiently nice, we know from Clairaut's Theorem (14.6.2) that $P_y = f_{xy} = f_{yx} = Q_x$. If we compute P_y and Q_x and find that they are not equal, then $\mathbf{F}$ is not conservative. If $P_y = Q_x$, then, again provided that $\mathbf{F}$ is sufficiently nice, we can be assured that $\mathbf{F}$ is conservative. Ultimately, what's important is that we be able to find f; as this amounts to finding anti-derivatives, we may not always succeed.

EXAMPLE 16.3.3 Find an f so that $\langle 3 + 2xy, x^2 - 3y^2\rangle = \nabla f$.

First, note that

$$\frac{\partial}{\partial y}(3 + 2xy) = 2x \qquad \text{and} \qquad \frac{\partial}{\partial x}(x^2 - 3y^2) = 2x,$$

so the desired f does exist. This means that $f_x = 3 + 2xy$, so that $f = 3x + x^2y + g(y)$; the first two terms are needed to get $3+2xy$, and the $g(y)$ could be any function of y, as it would disappear upon taking a derivative with respect to x. Likewise, since $f_y = x^2 - 3y^2$, $f = x^2y - y^3 + h(x)$. The question now becomes, is it possible to find $g(y)$ and $h(x)$ so that

$$3x + x^2y + g(y) = x^2y - y^3 + h(x),$$

and of course the answer is yes: $g(y) = -y^3$, $h(x) = 3x$. Thus, $f = 3x + x^2y - y^3$. □

We can test a vector field $\mathbf{F} = \langle P, Q, R\rangle$ in a similar way. Suppose that $\langle P, Q, R\rangle = \langle f_x, f_y, f_z\rangle$. If we temporarily hold z constant, then $f(x, y, z)$ is a function of x and y, and by Clairaut's Theorem $P_y = f_{xy} = f_{yx} = Q_x$. Likewise, holding y constant implies $P_z = f_{xz} = f_{zx} = R_x$, and with x constant we get $Q_z = f_{yz} = f_{zy} = R_y$. Conversely, if we find that $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$ then $\mathbf{F}$ is conservative.

Exercises 16.3.

1. Find an f so that $\nabla f = \langle 2x + y^2, 2y + x^2\rangle$, or explain why there is no such f. ⇒

2. Find an f so that $\nabla f = \langle x^3, -y^4\rangle$, or explain why there is no such f. ⇒

3. Find an f so that $\nabla f = \langle xe^y, ye^x\rangle$, or explain why there is no such f. ⇒

4. Find an f so that $\nabla f = \langle y\cos x, y\sin x\rangle$, or explain why there is no such f. ⇒

5. Find an f so that $\nabla f = \langle y\cos x, \sin x\rangle$, or explain why there is no such f. ⇒

6. Find an f so that $\nabla f = \langle x^2y^3, xy^4\rangle$, or explain why there is no such f. ⇒

7. Find an f so that $\nabla f = \langle yz, xz, xy\rangle$, or explain why there is no such f. ⇒

8. Evaluate $\displaystyle\int_C (10x^4 - 2xy^3)\,dx - 3x^2y^2\,dy$ where C is the part of the curve $x^5 - 5x^2y^2 - 7x^2 = 0$ from $(3, -2)$ to $(3, 2)$. ⇒

9. Let $\mathbf{F} = \langle yz, xz, xy\rangle$. Find the work done by this force field on an object that moves from $(1, 0, 2)$ to $(1, 2, 3)$. ⇒

10. Let $\mathbf{F} = \langle e^y, xe^y + \sin z, y\cos z\rangle$. Find the work done by this force field on an object that moves from $(0, 0, 0)$ to $(1, -1, 3)$. ⇒

11. Let

$$\mathbf{F} = \left\langle \frac{-x}{(x^2+y^2+z^2)^{3/2}}, \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \frac{-z}{(x^2+y^2+z^2)^{3/2}} \right\rangle.$$

Find the work done by this force field on an object that moves from $(1, 1, 1)$ to $(4, 5, 6)$. ⇒

16.4 Green's Theorem

We now come to the first of three important theorems that extend the Fundamental Theorem of Calculus to higher dimensions. (The Fundamental Theorem of Line Integrals has already done this in one way, but in that case we were still dealing with an essentially one-dimensional integral.) They all share with the Fundamental Theorem the following rather vague description: *To compute a certain sort of integral over a region, we may do a computation on the boundary of the region that involves one fewer integrations.*

Note that this does indeed describe the Fundamental Theorem of Calculus and the Fundamental Theorem of Line Integrals: to compute a single integral over an interval, we do a computation on the boundary (the endpoints) that involves one fewer integrations, namely, no integrations at all.

THEOREM 16.4.1 Green's Theorem If the vector field $\mathbf{F} = \langle P, Q\rangle$ and the region D are sufficiently nice, and if C is the boundary of D (C is a closed curve), then

$$\iint\limits_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \int_C P\, dx + Q\, dy,$$

provided the integration on the right is done counter-clockwise around C. □

To indicate that an integral $\int_C$ is being done over a closed curve in the counter-clockwise direction, we usually write $\oint_C$. We also use the notation ∂D to mean the boundary of D **oriented** in the counterclockwise direction. With this notation, $\oint_C = \int_{\partial D}$.

We already know one case, not particularly interesting, in which this theorem is true: If $\mathbf{F}$ is conservative, we know that the integral $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$, because any integral of a conservative vector field around a closed curve is zero. We also know in this case that $\partial P/\partial y = \partial Q/\partial x$, so the double integral in the theorem is simply the integral of the zero function, namely, 0. So in the case that $\mathbf{F}$ is conservative, the theorem says simply that $0 = 0$.

EXAMPLE 16.4.2 We illustrate the theorem by computing both sides of

$$\int_{\partial D} x^4\, dx + xy\, dy = \iint\limits_D y - 0\, dA,$$

where D is the triangular region with corners $(0,0)$, $(1,0)$, $(0,1)$.

Starting with the double integral:

$$\iint\limits_D y - 0\, dA = \int_0^1 \int_0^{1-x} y\, dy\, dx = \int_0^1 \frac{(1-x)^2}{2}\, dx = -\frac{(1-x)^3}{6}\bigg|_0^1 = \frac{1}{6}.$$

There is no single formula to describe the boundary of D, so to compute the left side directly we need to compute three separate integrals corresponding to the three sides of the triangle, and each of these integrals we break into two integrals, the "dx" part and the "dy" part. The three sides are described by $y = 0$, $y = 1 - x$, and $x = 0$. The integrals are then

$$\begin{aligned}\int_{\partial D} x^4\, dx + xy\, dy &= \int_0^1 x^4\, dx + \int_0^0 0\, dy + \int_1^0 x^4\, dx + \int_0^1 (1-y)y\, dy + \int_0^0 0\, dx + \int_1^0 0\, dy \\ &= \frac{1}{5} + 0 - \frac{1}{5} + \frac{1}{6} + 0 + 0 = \frac{1}{6}.\end{aligned}$$

Alternately, we could describe the three sides in vector form as $\langle t, 0\rangle$, $\langle 1-t, t\rangle$, and $\langle 0, 1-t\rangle$. Note that in each case, as t ranges from 0 to 1, we follow the corresponding side in the correct direction. Now

$$\begin{aligned}\int_{\partial D} x^4\,dx + xy\,dy &= \int_0^1 t^4 + t\cdot 0\,dt + \int_0^1 -(1-t)^4 + (1-t)t\,dt + \int_0^1 0 + 0\,dt\\ &= \int_0^1 t^4\,dt + \int_0^1 -(1-t)^4 + (1-t)t\,dt = \frac{1}{6}.\end{aligned}$$

□

In this case, none of the integrations are difficult, but the second approach is somewhat tedious because of the necessity to set up three different integrals. In different circumstances, either of the integrals, the single or the double, might be easier to compute. Sometimes it is worthwhile to turn a single integral into the corresponding double integral, sometimes exactly the opposite approach is best.

Here is a clever use of Green's Theorem: We know that areas can be computed using double integrals, namely,

$$\iint_D 1\,dA$$

computes the area of region D. If we can find P and Q so that $\partial Q/\partial x - \partial P/\partial y = 1$, then the area is also

$$\int_{\partial D} P\,dx + Q\,dy.$$

It is quite easy to do this: $P = 0, Q = x$ works, as do $P = -y, Q = 0$ and $P = -y/2, Q = x/2$.

EXAMPLE 16.4.3 An ellipse centered at the origin, with its two principal axes aligned with the x and y axes, is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

We find the area of the interior of the ellipse via Green's theorem. To do this we need a vector equation for the boundary; one such equation is $\langle a\cos t, b\sin t\rangle$, as t ranges from 0 to 2π. We can easily verify this by substitution:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{a^2\cos^2 t}{a^2} + \frac{b^2\sin^2 t}{b^2} = \cos^2 t + \sin^2 t = 1.$$

Let's consider the three possibilities for P and Q above: Using 0 and x gives

$$\oint_C 0\,dx + x\,dy = \int_0^{2\pi} a\cos(t) b\cos(t)\,dt = \int_0^{2\pi} ab\cos^2(t)\,dt.$$

Using $-y$ and 0 gives

$$\oint_C -y\,dx + 0\,dy = \int_0^{2\pi} -b\sin(t)(-a\sin(t))\,dt = \int_0^{2\pi} ab\sin^2(t)\,dt.$$

Finally, using $-y/2$ and $x/2$ gives

$$\begin{aligned}\oint_C -\frac{y}{2}\,dx + \frac{x}{2}\,dy &= \int_0^{2\pi} -\frac{b\sin(t)}{2}(-a\sin(t))\,dt + \frac{a\cos(t)}{2}(b\cos(t))\,dt \\ &= \int_0^{2\pi} \frac{ab\sin^2 t}{2} + \frac{ab\cos^2 t}{2}\,dt = \int_0^{2\pi} \frac{ab}{2}\,dt = \pi ab.\end{aligned}$$

The first two integrals are not particularly difficult, but the third is very easy, though the choice of P and Q seems more complicated. □

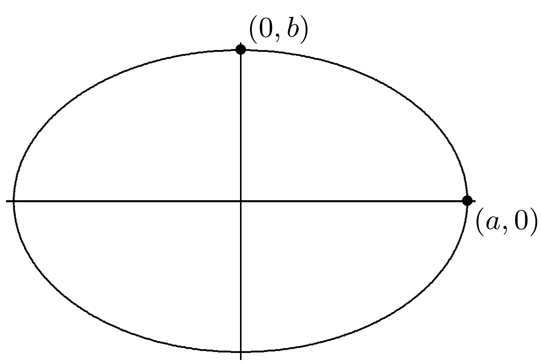


Figure 16.4.1 A "standard" ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Proof of Green's Theorem. We cannot here prove Green's Theorem in general, but we can do a special case. We seek to prove that

$$\oint_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA.$$

It is sufficient to show that

$$\oint_C P\,dx = \iint_D -\frac{\partial P}{\partial y}\,dA \qquad \text{and} \qquad \oint_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA,$$

which we can do if we can compute the double integral in both possible ways, that is, using $dA = dy\,dx$ and $dA = dx\,dy$.

For the first equation, we start with

$$\iint_D \frac{\partial P}{\partial y}\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}\,dy\,dx = \int_a^b P(x,g_2(x)) - P(x,g_1(x))\,dx.$$

Here we have simply used the ordinary Fundamental Theorem of Calculus, since for the inner integral we are integrating a derivative with respect to y: an antiderivative of $\partial P/\partial y$ with respect to y is simply $P(x,y)$, and then we substitute g_1 and g_2 for y and subtract.

Now we need to manipulate $\oint_C P\,dx$. The boundary of region D consists of 4 parts, given by the equations $y = g_1(x)$, $x = b$, $y = g_2(x)$, and $x = a$. On the portions $x = b$ and $x = a$, $dx = 0\,dt$, so the corresponding integrals are zero. For the other two portions, we use the parametric forms $x = t$, $y = g_1(t)$, $a \le t \le b$, and $x = t$, $y = g_2(t)$, letting t range from b to a, since we are integrating counter-clockwise around the boundary. The resulting integrals give us

$$\begin{aligned}\oint_C P\,dx = \int_a^b P(t,g_1(t))\,dt + \int_b^a P(t,g_2(t))\,dt &= \int_a^b P(t,g_1(t))\,dt - \int_a^b P(t,g_2(t))\,dt \\ &= \int_a^b P(t,g_1(t)) - P(t,g_2(t))\,dt\end{aligned}$$

which is the result of the double integral times -1, as desired.

The equation involving Q is essentially the same, and left as an exercise. ■

Exercises 16.4.

1. Compute $\displaystyle\int_{\partial D} 2y\,dx + 3x\,dy$, where D is described by $0 \le x \le 1$, $0 \le y \le 1$. ⇒

2. Compute $\displaystyle\int_{\partial D} xy\,dx + xy\,dy$, where D is described by $0 \le x \le 1$, $0 \le y \le 1$. ⇒

3. Compute $\displaystyle\int_{\partial D} e^{2x+3y}\,dx + e^{xy}\,dy$, where D is described by $-2 \le x \le 2$, $-1 \le y \le 1$. ⇒

4. Compute $\displaystyle\int_{\partial D} y\cos x\,dx + y\sin x\,dy$, where D is described by $0 \le x \le \pi/2$, $1 \le y \le 2$. ⇒

5. Compute $\displaystyle\int_{\partial D} x^2y\,dx + xy^2\,dy$, where D is described by $0 \le x \le 1$, $0 \le y \le x$. ⇒

6. Compute $\displaystyle\int_{\partial D} x\sqrt{y}\,dx + \sqrt{x+y}\,dy$, where D is described by $1 \le x \le 2$, $2x \le y \le 4$. ⇒

7. Compute $\displaystyle\int_{\partial D} (x/y)\,dx + (2+3x)\,dy$, where D is described by $1 \le x \le 2$, $1 \le y \le x^2$. ⇒

8. Compute $\displaystyle\int_{\partial D} \sin y\,dx + \sin x\,dy$, where D is described by $0 \le x \le \pi/2$, $x \le y \le \pi/2$. ⇒

9. Compute $\displaystyle\int_{\partial D} x\ln y\,dx$, where D is described by $1 \le x \le 2$, $e^x \le y \le e^{x^2}$. ⇒

10. Compute $\int_{\partial D} \sqrt{1+x^2}\,dy$, where D is described by $-1 \le x \le 1$, $x^2 \le y \le 1$. ⇒

11. Compute $\int_{\partial D} x^2y\,dx - xy^2\,dy$, where D is described by $x^2+y^2 \le 1$. ⇒

12. Compute $\int_{\partial D} y^3\,dx + 2x^3\,dy$, where D is described by $x^2+y^2 \le 4$. ⇒

13. Evaluate $\oint_C (y-\sin(x))\,dx + \cos(x)\,dy$, where C is the boundary of the triangle with vertices $(0,0)$, $(1,0)$, and $(1,2)$ oriented counter-clockwise. ⇒

14. Finish our proof of Green's Theorem by showing that $\oint_C Q\,dy = \iint\limits_D \frac{\partial Q}{\partial x}\,dA$.

16.5 Divergence and Curl

Divergence and curl are two measurements of vector fields that are very useful in a variety of applications. Both are most easily understood by thinking of the vector field as representing a flow of a liquid or gas; that is, each vector in the vector field should be interpreted as a velocity vector. Roughly speaking, divergence measures the tendency of the fluid to collect or disperse at a point, and curl measures the tendency of the fluid to swirl around the point. Divergence is a scalar, that is, a single number, while curl is itself a vector. The magnitude of the curl measures how much the fluid is swirling, the direction indicates the axis around which it tends to swirl. These ideas are somewhat subtle in practice, and are beyond the scope of this course. You can find additional information on the web, for example at `http://mathinsight.org/curl_idea` and `http://mathinsight.org/divergence_idea` and in many books including *Div, Grad, Curl, and All That: An Informal Text on Vector Calculus*, by H. M. Schey.

Recall that if f is a function, the gradient of f is given by

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle.$$

A useful mnemonic for this (and for the divergence and curl, as it turns out) is to let

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle,$$

that is, we pretend that ∇ is a vector with rather odd looking entries. Recalling that $\langle u,v,w\rangle a = \langle ua, va, wa\rangle$, we can then think of the gradient as

$$\nabla f = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle,$$

that is, we simply multiply the f into the vector.

The divergence and curl can now be defined in terms of this same odd vector ∇ by using the cross product and dot product. The divergence of a vector field $\mathbf{F} = \langle f, g, h \rangle$ is

$$\nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle f, g, h \rangle = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}.$$

The curl of $\mathbf{F}$ is

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle.$$

Here are two simple but useful facts about divergence and curl.

THEOREM 16.5.1 $\nabla \cdot (\nabla \times \mathbf{F}) = 0$. ■

In words, this says that the divergence of the curl is zero.

THEOREM 16.5.2 $\nabla \times (\nabla f) = \mathbf{0}$. ■

That is, the curl of a gradient is the zero vector. Recalling that gradients are conservative vector fields, this says that the curl of a conservative vector field is the zero vector. Under suitable conditions, it is also true that if the curl of $\mathbf{F}$ is $\mathbf{0}$ then $\mathbf{F}$ is conservative. (Note that this is exactly the same test that we discussed on page 427.)

EXAMPLE 16.5.3 Let $\mathbf{F} = \langle e^z, 1, xe^z \rangle$. Then $\nabla \times \mathbf{F} = \langle 0, e^z - e^z, 0 \rangle = \mathbf{0}$. Thus, $\mathbf{F}$ is conservative, and we can exhibit this directly by finding the corresponding f.

Since $f_x = e^z$, $f = xe^z + g(y,z)$. Since $f_y = 1$, it must be that $g_y = 1$, so $g(y,z) = y + h(z)$. Thus $f = xe^z + y + h(z)$ and

$$xe^z = f_z = xe^z + 0 + h'(z),$$

so $h'(z) = 0$, i.e., $h(z) = C$, and $f = xe^z + y + C$. □

We can rewrite Green's Theorem using these new ideas; these rewritten versions in turn are closer to some later theorems we will see.

Suppose we write a two dimensional vector field in the form $\mathbf{F} = \langle P, Q, 0 \rangle$, where P and Q are functions of x and y. Then

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \langle 0, 0, Q_x - P_y \rangle,$$

and so $(\nabla \times \mathbf{F}) \cdot \mathbf{k} = \langle 0, 0, Q_x - P_y \rangle \cdot \langle 0, 0, 1 \rangle = Q_x - P_y$. So Green's Theorem says

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \int_{\partial D} P\,dx + Q\,dy = \iint_D Q_x - P_y\,dA = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\,dA. \quad (16.5.1)$$

Roughly speaking, the right-most integral adds up the curl (tendency to swirl) at each point in the region; the left-most integral adds up the tangential components of the vector field around the entire boundary. Green's Theorem says these are equal, or roughly, that the sum of the "microscopic" swirls over the region is the same as the "macroscopic" swirl around the boundary.

Next, suppose that the boundary ∂D has a vector form $\mathbf{r}(t)$, so that $\mathbf{r}'(t)$ is tangent to the boundary, and $\mathbf{T} = \mathbf{r}'(t)/|\mathbf{r}'(t)|$ is the usual unit tangent vector. Writing $\mathbf{r} = \langle x(t), y(t)\rangle$ we get

$$\mathbf{T} = \frac{\langle x', y'\rangle}{|\mathbf{r}'(t)|}$$

and then

$$\mathbf{N} = \frac{\langle y', -x'\rangle}{|\mathbf{r}'(t)|}$$

is a unit vector perpendicular to $\mathbf{T}$, that is, a unit normal to the boundary. Now

$$\begin{aligned}\int_{\partial D} \mathbf{F}\cdot\mathbf{N}\,ds &= \int_{\partial D} \langle P, Q\rangle \cdot \frac{\langle y', -x'\rangle}{|\mathbf{r}'(t)|}|\mathbf{r}'(t)|dt = \int_{\partial D} Py'\,dt - Qx'\,dt\\ &= \int_{\partial D} P\,dy - Q\,dx = \int_{\partial D} -Q\,dx + P\,dy.\end{aligned}$$

So far, we've just rewritten the original integral using alternate notation. The last integral looks just like the right side of Green's Theorem (16.4.1) except that P and Q have traded places and Q has acquired a negative sign. Then applying Green's Theorem we get

$$\int_{\partial D} -Q\,dx + P\,dy = \iint\limits_D P_x + Q_y\,dA = \iint\limits_D \nabla\cdot\mathbf{F}\,dA.$$

Summarizing the long string of equalities,

$$\int_{\partial D} \mathbf{F}\cdot\mathbf{N}\,ds = \iint\limits_D \nabla\cdot\mathbf{F}\,dA. \qquad (16.5.2)$$

Roughly speaking, the first integral adds up the flow across the boundary of the region, from inside to out, and the second sums the divergence (tendency to spread) at each point in the interior. The theorem roughly says that the sum of the "microscopic" spreads is the same as the total spread across the boundary and out of the region.

Exercises 16.5.

1. Let $\mathbf{F} = \langle xy, -xy\rangle$ and let D be given by $0 \le x \le 1$, $0 \le y \le 1$. Compute $\int_{\partial D} \mathbf{F}\cdot d\mathbf{r}$ and $\int_{\partial D} \mathbf{F}\cdot\mathbf{N}\,ds$. ⇒

2. Let $\mathbf{F} = \langle ax^2, by^2\rangle$ and let D be given by $0 \le x \le 1$, $0 \le y \le 1$. Compute $\int_{\partial D} \mathbf{F}\cdot d\mathbf{r}$ and $\int_{\partial D} \mathbf{F}\cdot\mathbf{N}\,ds$. ⇒

3. Let $\mathbf{F} = \langle ay^2, bx^2\rangle$ and let D be given by $0 \le x \le 1$, $0 \le y \le x$. Compute $\int_{\partial D} \mathbf{F}\cdot d\mathbf{r}$ and $\int_{\partial D} \mathbf{F}\cdot\mathbf{N}\,ds$. ⇒

4. Let $\mathbf{F} = \langle \sin x\cos y, \cos x\sin y\rangle$ and let D be given by $0 \le x \le \pi/2$, $0 \le y \le x$. Compute $\int_{\partial D} \mathbf{F}\cdot d\mathbf{r}$ and $\int_{\partial D} \mathbf{F}\cdot\mathbf{N}\,ds$. ⇒

5. Let $\mathbf{F} = \langle y, -x\rangle$ and let D be given by $x^2 + y^2 \le 1$. Compute $\int_{\partial D} \mathbf{F}\cdot d\mathbf{r}$ and $\int_{\partial D} \mathbf{F}\cdot\mathbf{N}\,ds$. ⇒

6. Let $\mathbf{F} = \langle x, y\rangle$ and let D be given by $x^2 + y^2 \le 1$. Compute $\int_{\partial D} \mathbf{F}\cdot d\mathbf{r}$ and $\int_{\partial D} \mathbf{F}\cdot\mathbf{N}\,ds$. ⇒

7. Prove theorem 16.5.1.

8. Prove theorem 16.5.2.

9. If $\nabla\cdot\mathbf{F} = 0$, $\mathbf{F}$ is said to be **incompressible**. Show that any vector field of the form $\mathbf{F}(x, y, z) = \langle f(y, z), g(x, z), h(x, y)\rangle$ is incompressible. Give a non-trivial example.

16.6 Vector Functions for Surfaces

We have dealt extensively with vector equations for curves, $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$. A similar technique can be used to represent surfaces in a way that is more general than the equations for surfaces we have used so far. Recall that when we use $\mathbf{r}(t)$ to represent a curve, we imagine the vector $\mathbf{r}(t)$ with its tail at the origin, and then we follow the head of the arrow as t changes. The vector "draws" the curve through space as t varies.

Suppose we instead have a vector function of two variables,

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle.$$

As both u and v vary, we again imagine the vector $\mathbf{r}(u, v)$ with its tail at the origin, and its head sweeps out a surface in space. A useful analogy is the technology of CRT video screens, in which an electron gun fires electrons in the direction of the screen. The gun's direction sweeps horizontally and vertically to "paint" the screen with the desired image. In practice, the gun moves horizontally through an entire line, then moves vertically to the next line and repeats the operation. In the same way, it can be useful to imagine fixing a value of v and letting $\mathbf{r}(u, v)$ sweep out a curve as u changes. Then v can change a bit, and $\mathbf{r}(u, v)$ sweeps out a new curve very close to the first. Put enough of these curves together and they form a surface.

EXAMPLE 16.6.1 Consider the function $\mathbf{r}(u, v) = \langle v \cos u, v \sin u, v \rangle$. For a fixed value of v, as u varies from 0 to 2π, this traces a circle of radius v at height v above the x-y plane. Put lots and lots of these together,and they form a cone, as in figure 16.6.1. □

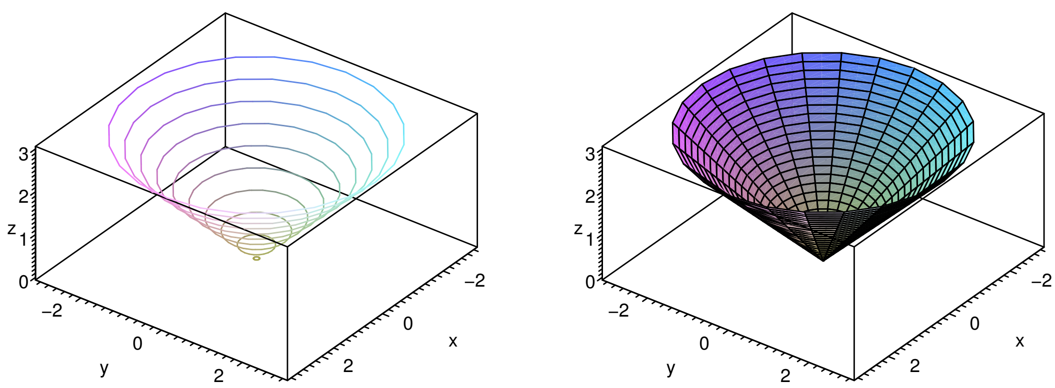


Figure 16.6.1 Tracing a surface.

EXAMPLE 16.6.2 Let $\mathbf{r} = \langle v \cos u, v \sin u, u \rangle$. If v is constant, the resulting curve is a helix (as in figure 13.1.1). If u is constant, the resulting curve is a straight line at height u in the direction u radians from the positive x axis. Note in figure 16.6.2 how the helixes and the lines both paint the same surface in a different way. □

This technique allows us to represent many more surfaces than previously.

EXAMPLE 16.6.3 The curve given by

$$\mathbf{r} = \langle (2 + \cos(3u/2)) \cos u, (2 + \cos(3u/2)) \sin u, \sin(3u/2) \rangle$$

is called a trefoil knot. Recall that from the vector equation of the curve we can compute the unit tangent $\mathbf{T}$, the unit normal $\mathbf{N}$, and the binormal vector $\mathbf{B} = \mathbf{T} \times \mathbf{N}$; you may want to review section 13.3. The binormal is perpendicular to both $\mathbf{T}$ and $\mathbf{N}$; one way to interpret this is that $\mathbf{N}$ and $\mathbf{B}$ define a plane perpendicular to $\mathbf{T}$, that is, perpendicular to the curve; since $\mathbf{N}$ and $\mathbf{B}$ are perpendicular to each other, they can function just as $\mathbf{i}$

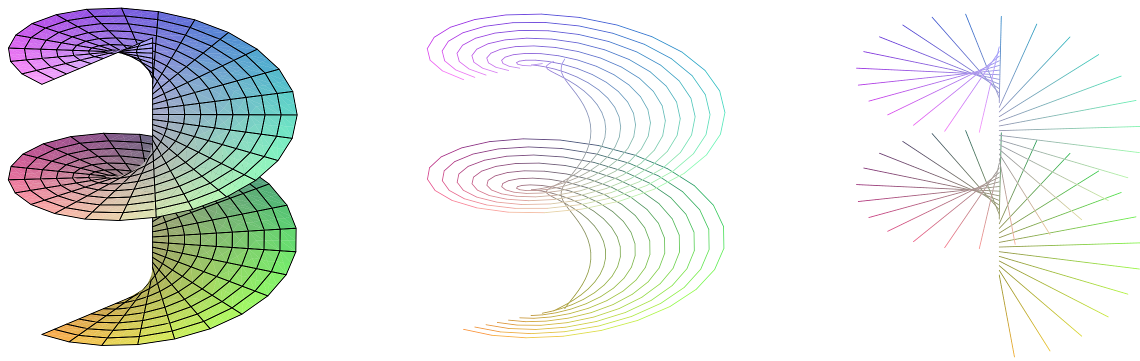


Figure 16.6.2 Tracing a surface. (AP)

and **j** do for the x-y plane. Of course, **N** and **B** are functions of u, changing as we move along the curve $\mathbf{r}(u)$. So, for example, $\mathbf{c}(u,v) = \mathbf{N}\cos v + \mathbf{B}\sin v$ is a vector equation for a unit circle in a plane perpendicular to the curve described by **r**, except that the usual interpretation of **c** would put its center at the origin. We can fix that simply by adding **c** to the original **r**: let $\mathbf{f} = \mathbf{r}(u) + \mathbf{c}(u,v)$. For a fixed u this draws a circle around the point $\mathbf{r}(u)$; as u varies we get a sequence of such circles around the curve **r**, that is, a tube of radius 1 with **r** at its center. We can easily change the radius; for example $\mathbf{r}(u) + a\mathbf{c}(u,v)$ gives the tube radius a; we can make the radius vary as we move along the curve with $\mathbf{r}(u) + g(u)\mathbf{c}(u,v)$, where $g(u)$ is a function of u. As shown in figure 16.6.3, it is hard to see that the plain knot is knotted; the tube makes the structure apparent. Of course, there is nothing special about the trefoil knot in this example; we can put a tube around (almost) any curve in the same way. □

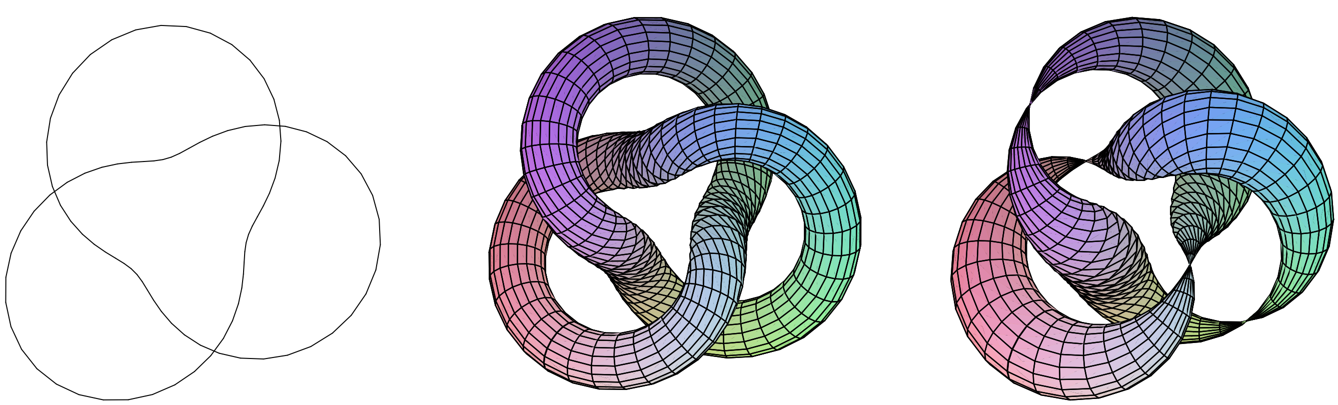


Figure 16.6.3 Tubes around a trefoil knot, with radius 1/2 and $3\cos(u)/4$. (AP)

We have previously examined surfaces given in the form $f(x,y)$. It is sometimes useful to represent such surfaces in the more general vector form, which is quite easy: $\mathbf{r}(u,v) = \langle u, v, f(u,v)\rangle$. The names of the variables are not important of course; instead of disguising x and y, we could simply write $\mathbf{r}(x,y) = \langle x, y, f(x,y)\rangle$.

We have also previously dealt with surfaces that are not functions of x and y; many of these are easy to represent in vector form. One common type of surface that cannot be represented as $z = f(x,y)$ is a surface given by an equation involving only x and y. For example, $x + y = 1$ and $y = x^2$ are "vertical" surfaces. For every point (x,y) in the plane that satisfies the equation, the point (x,y,z) is on the surface, for every value of z. Thus, a corresponding vector form for the surface is something like $\langle f(u), g(u), v\rangle$; for example, $x + y = 1$ becomes $\langle u, 1-u, v\rangle$ and $y = x^2$ becomes $\langle u, u^2, v\rangle$.

Yet another sort of example is the sphere, say $x^2+y^2+z^2 = 1$. This cannot be written in the form $z = f(x,y)$, but it is easy to write in vector form; indeed this particular surface is much like the cone, since it has circular cross-sections, or we can think of it as a tube around a portion of the z-axis, with a radius that varies depending on where along the axis we are. One vector expression for the sphere is $\langle \sqrt{1-v^2}\cos u, \sqrt{1-v^2}\sin u, v\rangle$— this emphasizes the tube structure, as it is naturally viewed as drawing a circle of radius $\sqrt{1-v^2}$ around the z-axis at height v. We could also take a cue from spherical coordinates, and write $\langle \sin u\cos v, \sin u\sin v, \cos u\rangle$, where in effect u and v are ϕ and θ in disguise.

It is quite simple in Sage to plot any surface for which you have a vector representation. Using different vector functions sometimes gives different looking plots, because Sage in effect draws the surface by holding one variable constant and then the other. For example, you might have noticed in figure 16.6.2 that the curves in the two right-hand graphs are superimposed on the left-hand graph; the graph of the surface is just the combination of the two sets of curves, with the spaces filled in with color.

Here's a simple but striking example: the plane $x + y + z = 1$ can be represented quite naturally as $\langle u, v, 1-u-v\rangle$. But we could also think of painting the same plane by choosing a particular point on the plane, say $(1,0,0)$, and then drawing circles or ellipses (or any of a number of other curves) as if that point were the origin in the plane. For example, $\langle 1 - v\cos u - v\sin u, v\sin u, v\cos u\rangle$ is one such vector function. Note that while it may not be obvious where this came from, it is quite easy to see that the sum of the x, y, and z components of the vector is always 1. Computer renderings of the plane using these two functions are shown in figure 16.6.4.

Suppose we know that a plane contains a particular point (x_0, y_0, z_0) and that two vectors $\mathbf{u} = \langle u_0, u_1, u_2\rangle$ and $\mathbf{v} = \langle v_0, v_1, v_2\rangle$ are parallel to the plane but not to each other. We know how to get an equation for the plane in the form $ax + by + cz = d$, by first computing $\mathbf{u}\times\mathbf{v}$. It's even easier to get a vector equation:

$$\mathbf{r}(u,v) = \langle x_0, y_0, z_0\rangle + u\mathbf{u} + v\mathbf{v}.$$

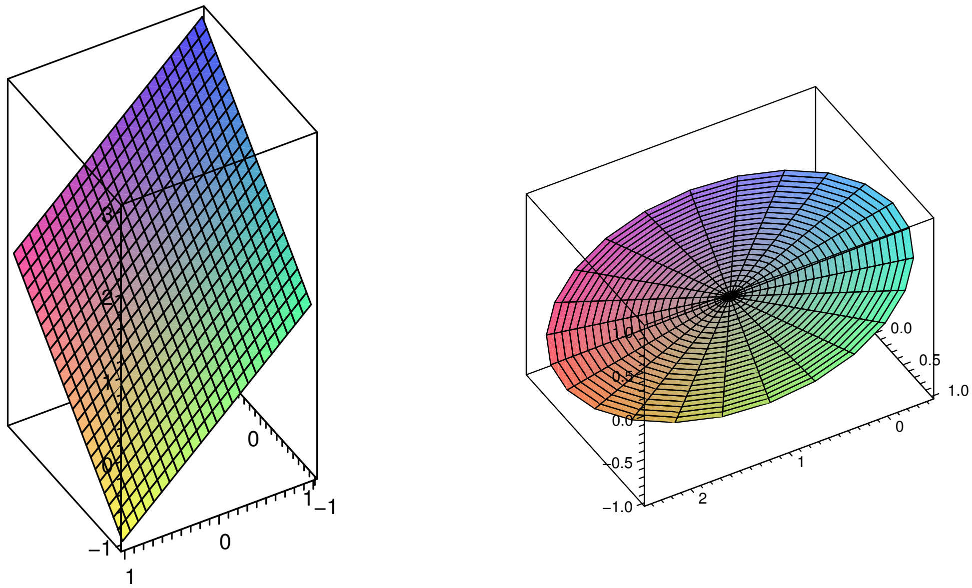


Figure 16.6.4 Two representations of the same plane. (AP)

The first vector gets to the point (x_0, y_0, z_0) and then by varying u and v, $u\mathbf{u} + v\mathbf{v}$ gets to every point in the plane.

Returning to $x + y + z = 1$, the points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ are all on the plane. By subtracting coordinates we see that $\langle -1, 0, 1\rangle$ and $\langle -1, 1, 0\rangle$ are parallel to the plane, so a third vector form for this plane is

$$\langle 1, 0, 0\rangle + u\langle -1, 0, 1\rangle + v\langle -1, 1, 0\rangle = \langle 1 - u - v, v, u\rangle.$$

This is clearly quite similar to the first form we found.

We have already seen (section 15.4) how to find the area of a surface when it is defined in the form $f(x, y)$. Finding the area when the surface is given as a vector function is very similar. Looking at the plots of surfaces we have just seen, it is evident that the two sets of curves that fill out the surface divide it into a grid, and that the spaces in the grid are approximately parallelograms. As before this is the key: we can write down the area of a typical little parallelogram and add them all up with an integral.

Suppose we want to approximate the area of the surface $\mathbf{r}(u, v)$ near $\mathbf{r}(u_0, v_0)$. The functions $\mathbf{r}(u, v_0)$ and $\mathbf{r}(u_0, v)$ define two curves that intersect at $\mathbf{r}(u_0, v_0)$. The derivatives of $\mathbf{r}$ give us vectors tangent to these two curves: $\mathbf{r}_u(u_0, v_0)$ and $\mathbf{r}_v(u_0, v_0)$, and then $\mathbf{r}_u(u_0, v_0)\,du$ and $\mathbf{r}_v(u_0, v_0)\,dv$ are two small tangent vectors, whose lengths can be used as the lengths of the sides of an approximating parallelogram. Finally, the area of this parallelogram is $|\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv$ and so the total surface area is

$$\int_a^b \int_c^d |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv.$$

EXAMPLE 16.6.4 We find the area of the surface $\langle v\cos u, v\sin u, u\rangle$ for $0 \le u \le \pi$ and $0 \le v \le 1$; this is a portion of the helical surface in figure 16.6.2. We compute $\mathbf{r}_u = \langle -v\sin u, v\cos u, 1\rangle$ and $\mathbf{r}_v = \langle \cos u, \sin u, 0\rangle$. The cross product of these two vectors is $\langle \sin u, -\cos u, v\rangle$ with length $\sqrt{1+v^2}$, and the surface area is

$$\int_0^\pi \int_0^1 \sqrt{1+v^2}\,dv\,du = \frac{\pi\sqrt{2}}{2} + \frac{\pi\ln(\sqrt{2}+1)}{2}.$$

□

Exercises 16.6.

1. Describe or sketch the surface with the given vector function.

- **a.** $\mathbf{r}(u,v) = \langle u+v, 3-v, 1+4u+5v\rangle$
- **b.** $\mathbf{r}(u,v) = \langle 2\sin u, 3\cos u, v\rangle$
- **c.** $\mathbf{r}(s,t) = \langle s, t, t^2 - s^2\rangle$
- **d.** $\mathbf{r}(s,t) = \langle s\sin 2t, s^2, s\cos 2t\rangle$

2. Find a vector function $\mathbf{r}(u,v)$ for the surface.

- **a.** The plane that passes through the point $(1,2,-3)$ and is parallel to the vectors $\langle 1,1,-1\rangle$ and $\langle 1,-1,1\rangle$.
- **b.** The lower half of the ellipsoid $2x^2+4y^2+z^2=1$.
- **c.** The part of the sphere of radius 4 centered at the origin that lies between the planes $z=-2$ and $z=2$.

3. Find the area of the portion of $x+2y+4z=10$ in the first octant. ⇒

4. Find the area of the portion of $2x+4y+z=0$ inside $x^2+y^2=1$. ⇒

5. Find the area of $z=x^2+y^2$ that lies below $z=1$. ⇒

6. Find the area of $z=\sqrt{x^2+y^2}$ that lies below $z=2$. ⇒

7. Find the area of the portion of $x^2+y^2+z^2=a^2$ that lies in the first octant. ⇒

8. Find the area of the portion of $x^2+y^2+z^2=a^2$ that lies above $x^2+y^2\le b^2$, $b\le a$. ⇒

9. Find the area of $z=x^2-y^2$ that lies inside $x^2+y^2=a^2$. ⇒

10. Find the area of $z=xy$ that lies inside $x^2+y^2=a^2$. ⇒

11. Find the area of $x^2+y^2+z^2=a^2$ that lies above the interior of the circle given in polar coordinates by $r=a\cos\theta$. ⇒

12. Find the area of the cone $z=k\sqrt{x^2+y^2}$ that lies above the interior of the circle given in polar coordinates by $r=a\cos\theta$. ⇒

13. Find the area of the plane $z=ax+by+c$ that lies over a region D with area A. ⇒

14. Find the area of the cone $z=k\sqrt{x^2+y^2}$ that lies over a region D with area A. ⇒

15. Find the area of the cylinder $x^2+z^2=a^2$ that lies inside the cylinder $x^2+y^2=a^2$. ⇒

16. The surface $f(x,y)$ can be represented with the vector function $\langle x,y,f(x,y)\rangle$. Set up the surface area integral using this vector function and compare to the integral of section 15.4.

16.7 Surface Integrals

In the integral for surface area,

$$\int_a^b \int_c^d |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv,$$

the integrand $|\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv$ is the area of a tiny parallelogram, that is, a very small surface area, so it is reasonable to abbreviate it dS; then a shortened version of the integral is

$$\iint_D 1 \cdot dS.$$

We have already seen that if D is a region in the plane, the area of D may be computed with

$$\iint_D 1 \cdot dA,$$

so this is really quite familiar, but the dS hides a little more detail than does dA.

Just as we can integrate functions $f(x, y)$ over regions in the plane, using

$$\iint_D f(x, y) \, dA,$$

so we can compute integrals over surfaces in space, using

$$\iint_D f(x, y, z) \, dS.$$

In practice this means that we have a vector function $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ for the surface, and the integral we compute is

$$\int_a^b \int_c^d f(x(u, v), y(u, v), z(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv.$$

That is, we express everything in terms of u and v, and then we can do an ordinary double integral.

EXAMPLE 16.7.1 Suppose a thin object occupies the upper hemisphere of $x^2 + y^2 + z^2 = 1$ and has density $\sigma(x, y, z) = z$. Find the mass and center of mass of the object. (Note that the object is just a thin shell; it does not occupy the interior of the hemisphere.)

We write the hemisphere as $\mathbf{r}(\phi,\theta) = \langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle$, $0 \le \phi \le \pi/2$ and $0 \le \theta \le 2\pi$. So $\mathbf{r}_\theta = \langle-\sin\theta\sin\phi, \cos\theta\sin\phi, 0\rangle$ and $\mathbf{r}_\phi = \langle\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi\rangle$. Then

$$\mathbf{r}_\theta \times \mathbf{r}_\phi = \langle-\cos\theta\sin^2\phi, -\sin\theta\sin^2\phi, -\cos\phi\sin\phi\rangle$$

and

$$|\mathbf{r}_\theta \times \mathbf{r}_\phi| = |\sin\phi| = \sin\phi,$$

since we are interested only in $0 \le \phi \le \pi/2$. Finally, the density is $z = \cos\phi$ and the integral for mass is

$$\int_0^{2\pi}\int_0^{\pi/2} \cos\phi\sin\phi\,d\phi\,d\theta = \pi.$$

By symmetry, the center of mass is clearly on the z-axis, so we only need to find the z-coordinate of the center of mass. The moment around the x-y plane is

$$\int_0^{2\pi}\int_0^{\pi/2} z\cos\phi\sin\phi\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/2} \cos^2\phi\sin\phi\,d\phi\,d\theta = \frac{2\pi}{3},$$

so the center of mass is at $(0,0,2/3)$. □

Now suppose that $\mathbf{F}$ is a vector field; imagine that it represents the velocity of some fluid at each point in space. We would like to measure how much fluid is passing through a surface D, the **flux** across D. As usual, we imagine computing the flux across a very small section of the surface, with area dS, and then adding up all such small fluxes over D with an integral. Suppose that vector $\mathbf{N}$ is a unit normal to the surface at a point; $\mathbf{F}\cdot\mathbf{N}$ is the scalar projection of $\mathbf{F}$ onto the direction of $\mathbf{N}$, so it measures how fast the fluid is moving across the surface. In one unit of time the fluid moving across the surface will fill a volume of $\mathbf{F}\cdot\mathbf{N}\,dS$, which is therefore the rate at which the fluid is moving across a small patch of the surface. Thus, the total flux across D is

$$\iint_D \mathbf{F}\cdot\mathbf{N}\,dS = \iint_D \mathbf{F}\cdot\,d\mathbf{S},$$

defining $d\mathbf{S} = \mathbf{N}\,dS$. As usual, certain conditions must be met for this to work out; chief among them is the nature of the surface. As we integrate over the surface, we must choose the normal vectors $\mathbf{N}$ in such a way that they point "the same way" through the surface. For example, if the surface is roughly horizontal in orientation, we might want to measure the flux in the "upwards" direction, or if the surface is closed, like a sphere, we might want to measure the flux "outwards" across the surface. In the first case we would choose $\mathbf{N}$ to have positive z component, in the second we would make sure that $\mathbf{N}$ points away from the origin. Unfortunately, there are surfaces that are not **orientable**: they have only one side, so that it is not possible to choose the normal vectors to point in the "same way" through the surface. The most famous such surface is the Möbius strip shown in figure 16.7.1. It is quite easy to make such a strip with a piece of paper and some tape. If you have never done this, it is quite instructive; in particular, you should draw a line down the center of the strip until you return to your starting point. No matter how unit normal vectors are assigned to the points of the Möbius strip, there will be normal vectors very close to each other pointing in opposite directions.

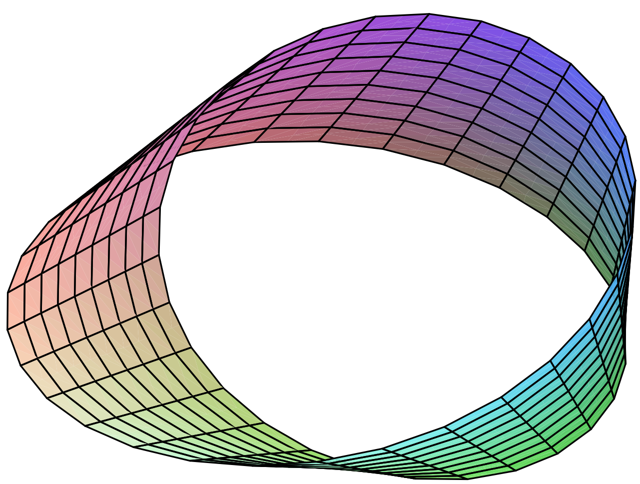


Figure 16.7.1 A Möbius strip. (AP)

Assuming that the quantities involved are well behaved, however, the flux of the vector field across the surface $\mathbf{r}(u,v)$ is

$$\iint_D \mathbf{F}\cdot\mathbf{N}\,dS = \iint_D \mathbf{F}\cdot\frac{\mathbf{r}_u\times\mathbf{r}_v}{|\mathbf{r}_u\times\mathbf{r}_v|}|\mathbf{r}_u\times\mathbf{r}_v|\,dA = \iint_D \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA.$$

In practice, we may have to use $\mathbf{r}_v\times\mathbf{r}_u$ or even something a bit more complicated to make sure that the normal vector points in the desired direction.

EXAMPLE 16.7.2 Compute the flux of $\mathbf{F}=\langle x,y,z^4\rangle$ across the cone $z=\sqrt{x^2+y^2}$, $0\le z\le 1$, in the downward direction.

We write the cone as a vector function: $\mathbf{r}=\langle v\cos u, v\sin u, v\rangle$, $0\le u\le 2\pi$ and $0\le v\le 1$. Then $\mathbf{r}_u=\langle -v\sin u, v\cos u, 0\rangle$ and $\mathbf{r}_v=\langle \cos u,\sin u,1\rangle$ and $\mathbf{r}_u\times\mathbf{r}_v=$

$\langle v\cos u, v\sin u, -v\rangle$. The third coordinate $-v$ is negative, which is exactly what we desire, that is, the normal vector points down through the surface. Then

$$\begin{aligned}\int_0^{2\pi}\int_0^1 \langle x,y,z^4\rangle\cdot\langle v\cos u, v\sin u, -v\rangle\,dv\,du &= \int_0^{2\pi}\int_0^1 xv\cos u + yv\sin u - z^4 v\,dv\,du\\ &= \int_0^{2\pi}\int_0^1 v^2\cos^2 u + v^2\sin^2 u - v^5\,dv\,du\\ &= \int_0^{2\pi}\int_0^1 v^2 - v^5\,dv\,du = \frac{\pi}{3}.\end{aligned}$$

□

Exercises 16.7.

1. Find the center of mass of an object that occupies the upper hemisphere of $x^2+y^2+z^2=1$ and has density x^2+y^2. ⇒

2. Find the center of mass of an object that occupies the surface $z=xy$, $0\le x\le 1$, $0\le y\le 1$ and has density $\sqrt{1+x^2+y^2}$. ⇒

3. Find the center of mass of an object that occupies the surface $z=\sqrt{x^2+y^2}$, $1\le z\le 4$ and has density x^2z. ⇒

4. Find the centroid of the surface of a right circular cone of height h and base radius r, not including the base. ⇒

5. Evaluate $\displaystyle\iint_D \langle 2,-3,4\rangle\cdot\mathbf{N}\,dS$, where D is given by $z=x^2+y^2$, $-1\le x\le 1$, $-1\le y\le 1$, oriented up. ⇒

6. Evaluate $\displaystyle\iint_D \langle x,y,3\rangle\cdot\mathbf{N}\,dS$, where D is given by $z=3x-5y$, $1\le x\le 2$, $0\le y\le 2$, oriented up. ⇒

7. Evaluate $\displaystyle\iint_D \langle x,y,-2\rangle\cdot\mathbf{N}\,dS$, where D is given by $z=1-x^2-y^2$, $x^2+y^2\le 1$, oriented up. ⇒

8. Evaluate $\displaystyle\iint_D \langle xy,yz,zx\rangle\cdot\mathbf{N}\,dS$, where D is given by $z=x+y^2+2$, $0\le x\le 1$, $x\le y\le 1$, oriented up. ⇒

9. Evaluate $\displaystyle\iint_D \langle e^x,e^y,z\rangle\cdot\mathbf{N}\,dS$, where D is given by $z=xy$, $0\le x\le 1$, $-x\le y\le x$, oriented up. ⇒

10. Evaluate $\displaystyle\iint_D \langle xz,yz,z\rangle\cdot\mathbf{N}\,dS$, where D is given by $z=a^2-x^2-y^2$, $x^2+y^2\le b^2$, oriented up. ⇒

11. A fluid has density 870 kg/m^3 and flows with velocity $\mathbf{v}=\langle z,y^2,x^2\rangle$, where distances are in meters and the components of $\mathbf{v}$ are in meters per second. Find the rate of flow outward through the portion of the cylinder $x^2+y^2=4$, $0\le z\le 1$ for which $y\ge 0$. ⇒

12. Gauss's Law says that the net charge, Q, enclosed by a closed surface, S, is

$$Q = \epsilon_0 \iint \mathbf{E} \cdot \mathbf{N}\, dS$$

where $\mathbf{E}$ is an electric field and ϵ_0 (the permittivity of free space) is a known constant; $\mathbf{N}$ is oriented outward. Use Gauss's Law to find the charge contained in the cube with vertices $(\pm 1, \pm 1, \pm 1)$ if the electric field is $\mathbf{E} = \langle x, y, z \rangle$. ⇒

16.8 Stokes's Theorem

Recall that one version of Green's Theorem (see equation 16.5.1) is

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA.$$

Here D is a region in the x-y plane and $\mathbf{k}$ is a unit normal to D at every point. If D is instead an orientable surface in space, there is an obvious way to alter this equation, and it turns out still to be true:

THEOREM 16.8.1 Stokes's Theorem Provided that the quantities involved are sufficiently nice, and in particular if D is orientable,

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS,$$

if ∂D is oriented counter-clockwise relative to $\mathbf{N}$. □

Note how little has changed: $\mathbf{k}$ becomes $\mathbf{N}$, a unit normal to the surface, and dA becomes dS, since this is now a general surface integral. The phrase "counter-clockwise relative to $\mathbf{N}$" means roughly that if we take the direction of $\mathbf{N}$ to be "up", then we go around the boundary counter-clockwise when viewed from "above". In many cases, this description is inadequate. A slightly more complicated but general description is this: imagine standing on the side of the surface considered positive; walk to the boundary and turn left. You are now following the boundary in the correct direction.

EXAMPLE 16.8.2 Let $\mathbf{F} = \langle e^{xy} \cos z, x^2 z, xy \rangle$ and the surface D be $x = \sqrt{1 - y^2 - z^2}$, oriented in the positive x direction. It quickly becomes apparent that the surface integral in Stokes's Theorem is intractable, so we try the line integral. The boundary of D is the unit circle in the y-z plane, $\mathbf{r} = \langle 0, \cos u, \sin u \rangle$, $0 \le u \le 2\pi$. The integral is

$$\int_0^{2\pi} \langle e^{xy} \cos z, x^2 z, xy \rangle \cdot \langle 0, -\sin u, \cos u \rangle\, du = \int_0^{2\pi} 0\, du = 0,$$

because $x = 0$. □

EXAMPLE 16.8.3 Consider the cylinder $\mathbf{r} = \langle \cos u, \sin u, v\rangle$, $0 \le u \le 2\pi$, $0 \le v \le 2$, oriented outward, and $\mathbf{F} = \langle y, zx, xy\rangle$. We compute

$$\iint_D \nabla \times \mathbf{F} \cdot \mathbf{N}\, dS = \int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$$

in two ways.

First, the double integral is

$$\int_0^{2\pi}\int_0^2 \langle 0, -\sin u, v-1\rangle \cdot \langle \cos u, \sin u, 0\rangle\, dv\, du = \int_0^{2\pi}\int_0^2 -\sin^2 u\, dv\, du = -2\pi.$$

The boundary consists of two parts, the bottom circle $\langle \cos t, \sin t, 0\rangle$, with t ranging from 0 to 2π, and $\langle \cos t, \sin t, 2\rangle$, with t ranging from 2π to 0. We compute the corresponding integrals and add the results:

$$\int_0^{2\pi} -\sin^2 t\, dt + \int_{2\pi}^0 -\sin^2 t + 2\cos^2 t = -\pi - \pi = -2\pi,$$

as before. □

An interesting consequence of Stokes's Theorem is that if D and E are two orientable surfaces with the same boundary, then

$$\iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS = \int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \int_{\partial E} \mathbf{F} \cdot d\mathbf{r} = \iint_E (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS.$$

Sometimes both of the integrals

$$\iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS \qquad \text{and} \qquad \int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$$

are difficult, but you may be able to find a second surface E so that

$$\iint_E (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS$$

has the same value but is easier to compute.

EXAMPLE 16.8.4 In example 16.8.2 the line integral was easy to compute. But we might also notice that another surface E with the same boundary is the flat disk $y^2+z^2 \le 1$, given by $\mathbf{r} = \langle 0, v\cos u, v\sin u\rangle$. The normal is $\mathbf{r_v} \times \mathbf{r_u} = \langle v, 0, 0\rangle$. We compute the curl:

$$\nabla \times \mathbf{F} = \langle x - x^2, -e^{xy}\sin z - y, 2xz - xe^{xy}\cos z\rangle.$$

Since $x = 0$ everywhere on the surface,

$$(\nabla \times \mathbf{F}) \cdot \mathbf{N} = \langle 0, -\sin z - y, 0\rangle \cdot \langle v, 0, 0\rangle = 0,$$

so the surface integral is

$$\iint_E 0\,dS = 0,$$

as before. In this case, of course, it is still somewhat easier to compute the line integral, avoiding $\nabla \times \mathbf{F}$ entirely. □

EXAMPLE 16.8.5 Let $\mathbf{F} = \langle -y^2, x, z^2\rangle$, and let the curve C be the intersection of the cylinder $x^2 + y^2 = 1$ with the plane $y + z = 2$, oriented counter-clockwise when viewed from above. We compute $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ in two ways.

First we do it directly: a vector function for C is $\mathbf{r} = \langle \cos u, \sin u, 2 - \sin u\rangle$, so $\mathbf{r}' = \langle -\sin u, \cos u, -\cos u\rangle$, and the integral is then

$$\int_0^{2\pi} y^2 \sin u + x\cos u - z^2 \cos u\,du = \int_0^{2\pi} \sin^3 u + \cos^2 u - (2 - \sin u)^2 \cos u\,du = \pi.$$

To use Stokes's Theorem, we pick a surface with C as the boundary; the simplest such surface is that portion of the plane $y + z = 2$ inside the cylinder. This has vector equation $\mathbf{r} = \langle v\cos u, v\sin u, 2 - v\sin u\rangle$. We compute $\mathbf{r}_u = \langle -v\sin u, v\cos u, -v\cos u\rangle$, $\mathbf{r}_v = \langle \cos u, \sin u, -\sin u\rangle$, and $\mathbf{r}_u \times \mathbf{r}_v = \langle 0, -v, -v\rangle$. To match the orientation of C we need to use the normal $\langle 0, v, v\rangle$. The curl of $\mathbf{F}$ is $\langle 0, 0, 1 + 2y\rangle = \langle 0, 0, 1 + 2v\sin u\rangle$, and the surface integral from Stokes's Theorem is

$$\int_0^{2\pi}\int_0^1 (1 + 2v\sin u)v\,dv\,du = \pi.$$

In this case the surface integral was more work to set up, but the resulting integral is somewhat easier. □

Proof of Stokes's Theorem. We can prove here a special case of Stokes's Theorem, which perhaps not too surprisingly uses Green's Theorem.

Suppose the surface D of interest can be expressed in the form $z = g(x, y)$, and let $\mathbf{F} = \langle P, Q, R\rangle$. Using the vector function $\mathbf{r} = \langle x, y, g(x, y)\rangle$ for the surface we get the surface integral

$$\iint_D \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_E \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle \cdot \langle -g_x, -g_y, 1\rangle \, dA$$
$$= \iint_E -R_y g_x + Q_z g_x - P_z g_y + R_x g_y + Q_x - P_y \, dA.$$

Here E is the region in the x-y plane directly below the surface D.

For the line integral, we need a vector function for ∂D. If $\langle x(t), y(t)\rangle$ is a vector function for ∂E then we may use $\mathbf{r}(t) = \langle x(t), y(t), g(x(t), y(t))\rangle$ to represent ∂D. Then

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \int_a^b P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt}\, dt = \int_a^b P\frac{dx}{dt} + Q\frac{dy}{dt} + R\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right) dt.$$

using the chain rule for dz/dt. Now we continue to manipulate this:

$$\int_a^b P\frac{dx}{dt} + Q\frac{dy}{dt} + R\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right) dt$$
$$= \int_a^b \left[\left(P + R\frac{\partial z}{\partial x}\right)\frac{dx}{dt} + \left(Q + R\frac{\partial z}{\partial y}\right)\frac{dy}{dt}\right] dt$$
$$= \int_{\partial E} \left(P + R\frac{\partial z}{\partial x}\right) dx + \left(Q + R\frac{\partial z}{\partial y}\right) dy,$$

which now looks just like the line integral of Green's Theorem, except that the functions P and Q of Green's Theorem have been replaced by the more complicated $P + R(\partial z/\partial x)$ and $Q + R(\partial z/\partial y)$. We can apply Green's Theorem to get

$$\int_{\partial E} \left(P + R\frac{\partial z}{\partial x}\right) dx + \left(Q + R\frac{\partial z}{\partial y}\right) dy = \iint_E \frac{\partial}{\partial x}\left(Q + R\frac{\partial z}{\partial y}\right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial z}{\partial x}\right) dA.$$

Now we can use the chain rule again to evaluate the derivatives inside this integral, and it becomes

$$\iint_E Q_x + Q_z g_x + R_x g_y + R_z g_x g_y + R g_{yx} - (P_y + P_z g_y + R_y g_x + R_z g_y g_x + R g_{xy})\, dA$$
$$= \iint_E Q_x + Q_z g_x + R_x g_y - P_y - P_z g_y - R_y g_x \, dA,$$

which is the same as the expression we obtained for the surface integral. ■

Exercises 16.8.

1. Let $\mathbf{F} = \langle z, x, y\rangle$. The plane $z = 2x + 2y - 1$ and the paraboloid $z = x^2 + y^2$ intersect in a closed curve. Stokes's Theorem implies that

$$\iint_{D_1} (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{D_2} (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS,$$

where the line integral is computed over the intersection C of the plane and the paraboloid, and the two surface integrals are computed over the portions of the two surfaces that have boundary C (provided, of course, that the orientations all match). Compute all three integrals. ⇒

2. Let D be the portion of $z = 1 - x^2 - y^2$ above the x-y plane, oriented up, and let $\mathbf{F} = \langle xy^2, -x^2y, xyz\rangle$. Compute $\iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS$. ⇒

3. Let D be the portion of $z = 2x + 5y$ inside $x^2 + y^2 = 1$, oriented up, and let $\mathbf{F} = \langle y, z, -x\rangle$. Compute $\int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$. ⇒

4. Compute $\oint_C x^2z\, dx + 3x\, dy - y^3\, dz$, where C is the unit circle $x^2 + y^2 = 1$ oriented counterclockwise. ⇒

5. Let D be the portion of $z = px + qy + r$ over a region in the x-y plane that has area A, oriented up, and let $\mathbf{F} = \langle ax + by + cz, ax + by + cz, ax + by + cz\rangle$. Compute $\int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$. ⇒

6. Let D be any surface and let $\mathbf{F} = \langle P(x), Q(y), R(z)\rangle$ (P depends only on x, Q only on y, and R only on z). Show that $\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = 0$.

7. Show that $\int_C f\nabla g + g\nabla f \cdot d\mathbf{r} = 0$, where $\mathbf{r}$ describes a closed curve C to which Stokes's Theorem applies. (See theorems 12.4.1 and 16.5.2.)

16.9 The Divergence Theorem

The third version of Green's Theorem (equation 16.5.2) we saw was:

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{N}\, ds = \iint_D \nabla \cdot \mathbf{F}\, dA.$$

With minor changes this turns into another equation, the Divergence Theorem:

THEOREM 16.9.1 Divergence Theorem Under suitable conditions, if E is a region of three dimensional space and D is its boundary surface, oriented outward, then

$$\iint_D \mathbf{F} \cdot \mathbf{N}\, dS = \iiint_E \nabla \cdot \mathbf{F}\, dV.$$

Proof. Again this theorem is too difficult to prove here, but a special case is easier. In the proof of a special case of Green's Theorem, we needed to know that we could describe the region of integration in both possible orders, so that we could set up one double integral using $dx\,dy$ and another using $dy\,dx$. Similarly here, we need to be able to describe the three-dimensional region E in different ways.

We start by rewriting the triple integral:

$$\iiint_E \nabla\cdot\mathbf{F}\,dV = \iiint_E (P_x+Q_y+R_z)\,dV = \iiint_E P_x\,dV + \iiint_E Q_y\,dV + \iiint_E R_z\,dV.$$

The double integral may be rewritten:

$$\iint_D \mathbf{F}\cdot\mathbf{N}\,dS = \iint_D (P\mathbf{i}+Q\mathbf{j}+R\mathbf{k})\cdot\mathbf{N}\,dS = \iint_D P\mathbf{i}\cdot\mathbf{N}\,dS + \iint_D Q\mathbf{j}\cdot\mathbf{N}\,dS + \iint_D R\mathbf{k}\cdot\mathbf{N}\,dS.$$

To prove that these give the same value it is sufficient to prove that

$$\begin{aligned}\iint_D P\mathbf{i}\cdot\mathbf{N}\,dS &= \iiint_E P_x\,dV,\\ \iint_D Q\mathbf{j}\cdot\mathbf{N}\,dS &= \iiint_E Q_y\,dV, \text{ and}\\ \iint_D R\mathbf{k}\cdot\mathbf{N}\,dS &= \iiint_E R_z\,dV.\end{aligned} \tag{16.9.1}$$

Not surprisingly, these are all pretty much the same; we'll do the first one.

We set the triple integral up with dx innermost:

$$\iiint_E P_x\,dV = \iint_B \int_{g_1(y,z)}^{g_2(y,z)} P_x\,dx\,dA = \iint_B P(g_2(y,z),y,z) - P(g_1(y,z),y,z)\,dA,$$

where B is the region in the y-z plane over which we integrate. The boundary surface of E consists of a "top" $x = g_2(y,z)$, a "bottom" $x = g_1(y,z)$, and a "wrap-around side" that is vertical to the y-z plane. To integrate over the entire boundary surface, we can integrate over each of these (top, bottom, side) and add the results. Over the side surface, the vector $\mathbf{N}$ is perpendicular to the vector $\mathbf{i}$, so

$$\iint_{\text{side}} P\mathbf{i}\cdot\mathbf{N}\,dS = \iint_{\text{side}} 0\,dS = 0.$$

Thus, we are left with just the surface integral over the top plus the surface integral over the bottom. For the top, we use the vector function $\mathbf{r} = \langle g_2(y,z), y, z\rangle$ which gives

$\mathbf{r}_y \times \mathbf{r}_z = \langle 1, -g_{2y}, -g_{2z} \rangle$; the dot product of this with $\mathbf{i} = \langle 1, 0, 0 \rangle$ is 1. Then

$$\iint_{\text{top}} P\mathbf{i} \cdot \mathbf{N}\, dS = \iint_B P(g_2(y,z), y, z)\, dA.$$

In almost identical fashion we get

$$\iint_{\text{bottom}} P\mathbf{i} \cdot \mathbf{N}\, dS = -\iint_B P(g_1(y,z), y, z)\, dA,$$

where the negative sign is needed to make $\mathbf{N}$ point in the negative x direction. Now

$$\iint_D P\mathbf{i} \cdot \mathbf{N}\, dS = \iint_B P(g_2(y,z), y, z)\, dA - \iint_B P(g_1(y,z), y, z)\, dA,$$

which is the same as the value of the triple integral above. ■

EXAMPLE 16.9.2 Let $\mathbf{F} = \langle 2x, 3y, z^2 \rangle$, and consider the three-dimensional volume inside the cube with faces parallel to the principal planes and opposite corners at $(0,0,0)$ and $(1,1,1)$. We compute the two integrals of the divergence theorem.

The triple integral is the easier of the two:

$$\int_0^1 \int_0^1 \int_0^1 2 + 3 + 2z\, dx\, dy\, dz = 6.$$

The surface integral must be separated into six parts, one for each face of the cube. One face is $z = 0$ or $\mathbf{r} = \langle u, v, 0 \rangle$, $0 \le u, v \le 1$. Then $\mathbf{r}_u = \langle 1, 0, 0 \rangle$, $\mathbf{r}_v = \langle 0, 1, 0 \rangle$, and $\mathbf{r}_u \times \mathbf{r}_v = \langle 0, 0, 1 \rangle$. We need this to be oriented downward (out of the cube), so we use $\langle 0, 0, -1 \rangle$ and the corresponding integral is

$$\int_0^1 \int_0^1 -z^2\, du\, dv = \int_0^1 \int_0^1 0\, du\, dv = 0.$$

Another face is $y = 1$ or $\mathbf{r} = \langle u, 1, v \rangle$. Then $\mathbf{r}_u = \langle 1, 0, 0 \rangle$, $\mathbf{r}_v = \langle 0, 0, 1 \rangle$, and $\mathbf{r}_u \times \mathbf{r}_v = \langle 0, -1, 0 \rangle$. We need a normal in the positive y direction, so we convert this to $\langle 0, 1, 0 \rangle$, and the corresponding integral is

$$\int_0^1 \int_0^1 3y\, du\, dv = \int_0^1 \int_0^1 3\, du\, dv = 3.$$

The remaining four integrals have values 0, 0, 2, and 1, and the sum of these is 6, in agreement with the triple integral. □

EXAMPLE 16.9.3 Let $\mathbf{F} = \langle x^3, y^3, z^2\rangle$, and consider the cylindrical volume $x^2+y^2 \le 9$, $0 \le z \le 2$. The triple integral (using cylindrical coordinates) is

$$\int_0^{2\pi}\int_0^3\int_0^2 (3r^2+2z)r\,dz\,dr\,d\theta = 279\pi.$$

For the surface we need three integrals. The top of the cylinder can be represented by $\mathbf{r} = \langle v\cos u, v\sin u, 2\rangle$; $\mathbf{r}_u \times \mathbf{r}_v = \langle 0, 0, -v\rangle$, which points down into the cylinder, so we convert it to $\langle 0, 0, v\rangle$. Then

$$\int_0^{2\pi}\int_0^3 \langle v^3\cos^3 u, v^3\sin^3 u, 4\rangle \cdot \langle 0, 0, v\rangle\,dv\,du = \int_0^{2\pi}\int_0^3 4v\,dv\,du = 36\pi.$$

The bottom is $\mathbf{r} = \langle v\cos u, v\sin u, 0\rangle$; $\mathbf{r}_u \times \mathbf{r}_v = \langle 0, 0, -v\rangle$ and

$$\int_0^{2\pi}\int_0^3 \langle v^3\cos^3 u, v^3\sin^3 u, 0\rangle \cdot \langle 0, 0, -v\rangle\,dv\,du = \int_0^{2\pi}\int_0^3 0\,dv\,du = 0.$$

The side of the cylinder is $\mathbf{r} = \langle 3\cos u, 3\sin u, v\rangle$; $\mathbf{r}_u \times \mathbf{r}_v = \langle 3\cos u, 3\sin u, 0\rangle$ which does point outward, so

$$\begin{aligned}\int_0^{2\pi}\int_0^2 &\langle 27\cos^3 u, 27\sin^3 u, v^2\rangle \cdot \langle 3\cos u, 3\sin u, 0\rangle\,dv\,du\\ &= \int_0^{2\pi}\int_0^2 81\cos^4 u + 81\sin^4 u\,dv\,du = 243\pi.\end{aligned}$$

The total surface integral is thus $36\pi + 0 + 243\pi = 279\pi$. □

Exercises 16.9.

1. Using $\mathbf{F} = \langle 3x, y^3, -2z^2\rangle$ and the region bounded by $x^2+y^2 = 9$, $z = 0$, and $z = 5$, compute both integrals from the Divergence Theorem. ⇒
2. Let E be the volume described by $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$, and $\mathbf{F} = \langle x^2, y^2, z^2\rangle$. Compute $\displaystyle\iint_{\partial E} \mathbf{F}\cdot\mathbf{N}\,dS$. ⇒
3. Let E be the volume described by $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$, and $\mathbf{F} = \langle 2xy, 3xy, ze^{x+y}\rangle$. Compute $\displaystyle\iint_{\partial E} \mathbf{F}\cdot\mathbf{N}\,dS$. ⇒
4. Let E be the volume described by $0 \le x \le 1$, $0 \le y \le x$, $0 \le z \le x+y$, and $\mathbf{F} = \langle x, 2y, 3z\rangle$. Compute $\displaystyle\iint_{\partial E} \mathbf{F}\cdot\mathbf{N}\,dS$. ⇒

5. Let E be the volume described by $x^2 + y^2 + z^2 \le 4$, and $\mathbf{F} = \langle x^3, y^3, z^3 \rangle$. Compute $\iint\limits_{\partial E} \mathbf{F} \cdot \mathbf{N}\, dS$. ⇒

6. Let E be the hemisphere described by $0 \le z \le \sqrt{1 - x^2 - y^2}$, and $\mathbf{F} = \langle \sqrt{x^2 + y^2 + z^2}, \sqrt{x^2 + y^2 + z^2}, \sqrt{x^2 + y^2 + z^2} \rangle$. Compute $\iint\limits_{\partial E} \mathbf{F} \cdot \mathbf{N}\, dS$. ⇒

7. Let E be the volume described by $x^2 + y^2 \le 1$, $0 \le z \le 4$, and $\mathbf{F} = \langle xy^2, yz, x^2 z \rangle$. Compute $\iint\limits_{\partial E} \mathbf{F} \cdot \mathbf{N}\, dS$. ⇒

8. Let E be the solid cone above the x-y plane and inside $z = 1 - \sqrt{x^2 + y^2}$, and $\mathbf{F} = \langle x \cos^2 z, y \sin^2 z, \sqrt{x^2 + y^2} z \rangle$. Compute $\iint\limits_{\partial E} \mathbf{F} \cdot \mathbf{N}\, dS$. ⇒

9. Prove the other two equations in the display 16.9.1.

10. Suppose D is a closed surface, and that D and F are sufficiently nice. Show that

$$\iint\limits_{D} (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS = 0$$

where $\mathbf{N}$ is the outward pointing unit normal.

11. Suppose D is a closed surface, D is sufficiently nice, and $F = \langle a, b, c \rangle$ is a constant vector field. Show that

$$\iint\limits_{D} \mathbf{F} \cdot \mathbf{N}\, dS = 0$$

where $\mathbf{N}$ is the outward pointing unit normal.

12. We know that the volume of a region E may often be computed as $\iiint\limits_{E} dx\, dy\, dz$. Show that this volume may also be computed as $\frac{1}{3} \iint\limits_{\partial E} \langle x, y, z \rangle \cdot \mathbf{N}\, dS$ where $\mathbf{N}$ is the outward pointing unit normal to ∂E.

2

Fourier Series and Fourier Transform

2.1 INTRODUCTION

Fourier series is used to get frequency spectrum of a time-domain signal, when signal is a periodic function of time. We have seen that the sum of two sinusoids is periodic provided their frequencies are integer multiple of a fundamental frequency, w_0.

2.2 TRIGONOMETRIC FOURIER SERIES

Consider a signal $x(t)$, a sum of sine and cosine function whose frequencies are integral multiple of w_0

$$x(t) = a_0 + a_1 \cos(w_0 t) + a_2 \cos(2w_0 t) + \cdots$$
$$b_1 \sin(w_0 t) + b_2 \sin(2w_0 t) + \cdots$$
$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nw_0 t) + b_n \sin(nw_0 t)) \tag{1}$$

$a_0,\ a_1, \ldots,\ b_1,\ b_2, \ldots$ are constants and w_0 is the fundamental frequency.

Evaluation of Fourier Coefficients

To evaluate a_0 we shall integrate both sides of eqn. (1) over one period $(t_0, t_0 + T)$ of $x(t)$ at an arbitrary time t_0

$$\int_{t_0}^{t_0+T} x(t)dt = \int_{t_0}^{t_0+T} a_0 dt + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T} \cos(nw_0 t)dt + \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T} \sin(nw_0 t)dt$$

Since $\int_{t_0}^{t_0+T} \cos(nw_0 dt) = 0$

$$\int_{t_0}^{t_0+T} \sin(nw_0 dt) = 0$$

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t)dt \tag{2}$$

To evaluate a_n and b_n, we use the following result:

$$\int_{t_0}^{t_0+T} \cos(nw_0 t) \cos(mw_0 t)dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \neq 0 \end{cases}$$

Multiply eqn. (1) by sin (mw_0t) and integrate over one period

$$\int_{t_0}^{t_0+T} x(t)\sin(mw_0t)dt = a_0\int_{t_0}^{t_0+T}\sin(mw_0t)dt + \sum_{n=1}^{\infty} a_n\int_{t_0}^{t_0+T}\cos(nw_0t)\sin(mw_0t)dt + \sum_{n=1}^{\infty} b_n\int_{t_0}^{t_0+T}\sin(mw_0t)\sin(nw_0t)dt$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin(nw_0t)dt \tag{4}$$

Example 1:

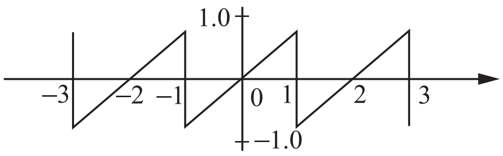


Fig. 2.1.

$$T \to -1 \text{ to } 1 \qquad T = 2 \qquad w_0 = \pi \qquad x(t) = t, -1 < t < 1$$

$$a_0 = \frac{1}{2}\int_{-1}^{1} t\,dt = \frac{1}{4}(1-1) = 0$$

$$a_n = 0$$

$$b_n = \int_{-1}^{1} t\sin\pi nt dt = \left[\frac{-t\cos\pi nt}{n\pi} - \frac{\cos\pi nt}{n\pi}\right]_{-1}^{1}$$

$$= \frac{-1}{n\pi}[t\cos\pi nt + \cos\pi nt]_{-1}^{1} = -\frac{1}{n\pi}[2\cos\pi + \cos\pi - \cos\pi]$$

$$b_n = \frac{-2}{n\pi}\cos n\pi = \frac{2}{\pi}\left[\frac{-(-1)^n}{n}\right]$$

b_1	b_2	b_3	b_4	b_5	b_6
$\frac{2}{\pi}$	$\frac{-2}{2\pi}$	$\frac{2}{3\pi}$	$\frac{-2}{4\pi}$	$\frac{2}{5\pi}$	$\frac{-2\cdots}{6\pi}$

$$x(t) = \sum_{n=1}^{\infty}\frac{2}{\pi}\left[\frac{-(-1)^n}{n}\right]\sin n\pi t$$

$$= \frac{2}{\pi}\left[\sin\pi t - \frac{1}{2}\sin 2\pi t + \frac{1}{3}\sin 3\pi t - \frac{1}{4}\sin 4\pi t + \cdots\right]$$

Example 2:

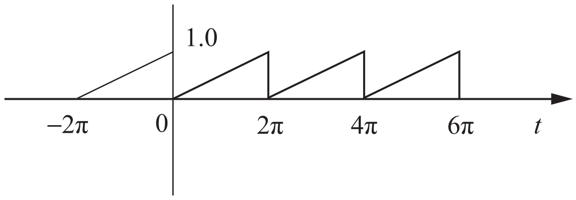


Fig. 2.2.

$$x(t) = \frac{t}{2\pi} \qquad T = 2\pi \qquad w_0 = \frac{2\pi}{T} = 1$$

$$a_0 = \frac{1}{T}\int_0^{2\pi} x(t)dt = \frac{1}{4\pi^2}\left[\frac{1}{2}t^2\right]_0^{2\pi} = \frac{1}{2}$$

$$a_n = \frac{2}{4\pi^2}\int_0^{2\pi} t\cos ntdt = \frac{1}{2\pi^2}\left[\frac{t\sin t}{n} + \frac{\sin nt}{n}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi^2}\left[\frac{2\pi\sin 2n\pi}{n} + \frac{\sin 2n\pi}{n}\right] = 0$$

$$b_n = \frac{2}{4\pi^2}\int_0^{2\pi} t\sin ntdt = \frac{-1}{2\pi^2}\left[\frac{t\cos nt}{n} + \frac{\cos nt}{n}\right]_0^{2\pi}$$

$$= \frac{-1}{2\pi^2}\left[\frac{2\pi\cos 2n\pi}{n} + \frac{\cos 2n\pi}{n} - \frac{1}{n}\right]$$

$$b_n = \frac{-1}{n\pi}$$

$$x(t) = \frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{-1}{n\pi}\right)\sin nt = \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cos\left(nt + \pi/2\right)$$

$$= \frac{1}{2} - \frac{1}{\pi}\left[\sin t + \frac{\sin 2t}{2} + \frac{\sin 3t}{3} + \cdots\right]$$

Example 3:

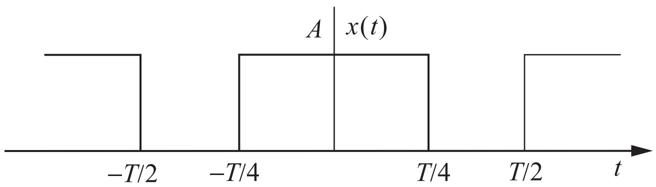


Fig. 2.3. Rectangular waveform

Figure shows a periodic rectangular waveform which is symmetrical to the vertical axis. Obtain its F.S. representation.

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos\, nw_0t + b_n \sin\, nw_0t)$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\,(nw_0t) \quad bn = 0$$

$$x(t) = 0 \quad \text{for } \frac{-T}{2} < t < \frac{-T}{4}$$

$$+A \quad \text{for } \frac{-T}{4} < t < \frac{T}{4}$$

$$0 \quad \text{for } \frac{T}{4} < t < \frac{T}{2}$$

$$a_0 = \frac{1}{T}\int_{-T/4}^{T/4} A dt = \frac{A}{2}$$

$$a_n = \frac{2}{T}\int_{-T/4}^{T/4} A \cos\,(nw_0t)dt = \frac{2A}{Tnw_0}\left[\sin\, nw_0\frac{T}{4} + \sin\, nw_0\frac{T}{4}\right]$$

$$a_n = \frac{4A}{2\pi n}\sin\left(\frac{n\pi}{2}\right) = \frac{2A}{\pi n}\sin\left(\frac{n\pi}{2}\right) \quad w_0 = \frac{2\pi}{T}$$

$$a_1 = \frac{4A}{2\pi} = \frac{2A}{\pi}$$

$$a_2 = 0$$

$$a_3 = \frac{2A}{3\pi}\sin\frac{3\pi}{2} = \frac{2A}{3\pi}(-1) = \frac{-2A}{3\pi}$$

$$x(t) = \frac{A}{2} + \frac{2A}{\pi}\left(\cos\, w_0t - \frac{1}{3}\cos\, 3w_0t + \frac{1}{5}\cos\, 5w_0t + \cdots\right)$$

Example 4: Find the trigonometric Fourier series for the periodic signal $x(t)$.

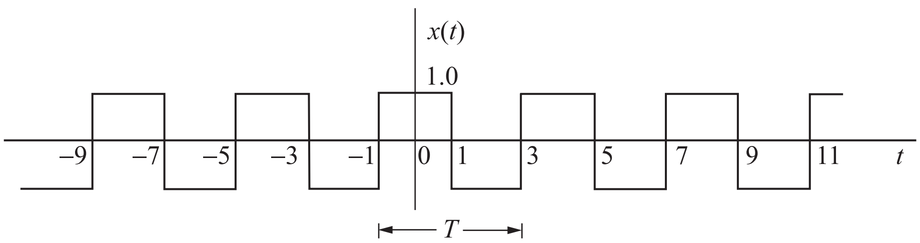


Fig. 2.4.

SOLUTION:

$$b_n = 0 \quad x(t) = \begin{cases} 1 & -1 < t < 1 \\ -1 & 1 < t < 3 \end{cases}$$

$$a_0 = \frac{1}{T}\int_{-1}^{3} x(t)dt = \frac{1}{T}\left[\int_{-1}^{1} dt + \int_{t}^{3}(-1)dt\right] \qquad T = 4$$

$$= \frac{1}{T}[2-2] = 0 \qquad \therefore \quad w_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$a_n = \frac{2}{T}\left[\int_{-1}^{1} \cos\,(nw_0t)dt + \int_{1}^{3} \cos\,(nw_0t)dt\right]$$

$$= \frac{2}{2\pi n}\left\{\left[2\sin\frac{\pi n}{2}\right] - \left[\sin\frac{3n\pi}{2} - \sin\frac{n\pi}{2}\right]\right\}$$

$$= \frac{1}{n\pi}\left[3\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{2}\right] \qquad \sin\frac{3n\pi}{2} = \sin\left(\pi + \frac{n\pi}{2}\right) = -\sin\frac{n\pi}{2}$$

$$a_n = \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$a_n = \begin{cases} 0 & n = \text{even} \\ \dfrac{4}{n\pi} & n = 1, 5, 9, 13 \\ \dfrac{-4}{n\pi} & n = 3, 7, 11, 15 \end{cases}$$

$$x(t) = \frac{4}{\pi}\cos\left(\frac{\pi}{2}t\right) - \frac{4}{3\pi}\cos\left(\frac{3\pi}{2}t\right) + \frac{4}{5\pi}\cos\left(\frac{5\pi}{2}t\right) - \frac{4}{7\pi}\cos\left(\frac{7\pi}{2}t\right) + \cdots$$

$$x(t) = \frac{4}{\pi}\left[\cos\left(\frac{\pi}{2}t\right) - \frac{1}{3}\cos\left(\frac{3\pi}{2}t\right) + \frac{1}{5}\cos\left(\frac{5\pi}{2}t\right)\cdots\right]$$

Example 5: Find the F.S.C. for the continuous-time periodic signal

$$x(t) = 1.5 \qquad 0 \le t < 1$$
$$= -1.5 \qquad 1 \le t < 2$$

with fundamental freq. $w_0 = \pi$

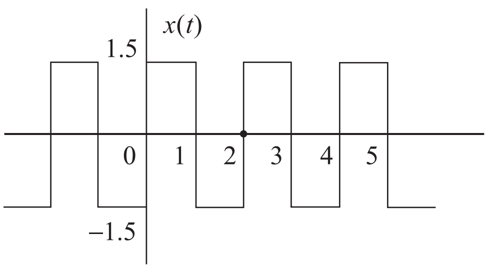


Fig. 2.5.

SOLUTION:

$$T = \frac{2\pi}{w_0} = 2,\ w_0 = \pi$$

$$a_0 = a_n = 0$$

$$b_n = \int_0^1 1.5 \sin n\pi t\, dt - \int_1^2 1.5 \sin n\pi t\, dt$$

$$= \frac{1.5}{n\pi}\left\{ [-\cos n\pi + 1] + [\cos 2n\pi - \cos n\pi] \right\}$$

$$b_n = \frac{3}{n\pi}[1 - \cos n\pi]$$

$$x(t) = \frac{3}{\pi}\left[2 \sin \pi t + \frac{2}{3} \sin 3\pi t + \frac{2}{5} \sin 5\pi t + \cdots\right]$$

$$\frac{6}{\pi}\left[\sin \pi t + \frac{1}{3} \sin 3\pi t + \frac{1}{5} \sin 5\pi t + \cdots\right]$$

$$C_0 = \frac{1}{2}\left[\int_0^1 1.5 dt - 1.5 \int_1^2 dt\right] = 0$$

OR

By using complex exponential Fourier series

$$C_n = \frac{1}{2}\left[\int_0^1 1.5 e^{-jn\pi t} dt - 1.5 \int_1^2 e^{-jn\pi t} dt\right]$$

$$C_n = \frac{3}{-4jn\pi}\left[e^{-jn\pi t}\Big|_0^1 - e^{-jn\pi t}\Big|_1^2 \right]$$

$$= \frac{-3}{4jn\pi}\left[e^{-jn\pi} - 1 - e^{-j2n\pi} + e^{-jn\pi}\right]$$

$$= \frac{3}{2jn\pi}\left[1 - e^{-jn\pi}\right] = \frac{3}{2jn\pi}[1 - \cos n\pi]$$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{-jn\pi t}$$

$$\sum_{n=-\infty}^{\infty} \frac{3}{2jn\pi}\left[1 - e^{-jn\pi}\right] e^{jn\pi t}$$

$$= \sum_{n=-\infty}^{\infty} \frac{3}{2jn\pi}\left[e^{jn\pi t} - e^{jn\pi t} \cos \pi n\right]$$

for $n = 1$

$$= \frac{A}{2\pi}\int_0^{\pi} \sin t \sin t dt = \frac{A}{2\pi}\int_0^{\pi}(1-\cos 2t)dt$$

$$= \frac{A}{2\pi}[\pi] = \frac{A}{2}$$

When n is even

$$= \frac{A}{2\pi}\left[\frac{2}{n+1} - \frac{2}{1-n}\right] = \frac{2A}{\pi(1-n^2)}$$

Example 7:

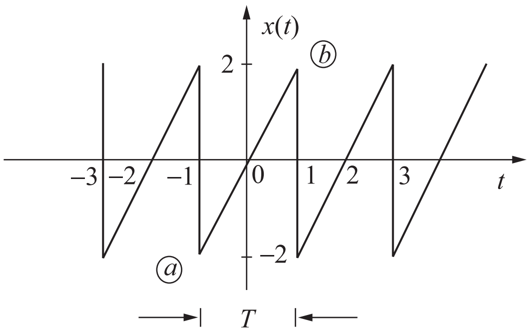


Fig. 2.7.

SOLUTION:

$$T = 2 \quad w_0 = \frac{2\pi}{T} = \pi$$

$$x(t) = \begin{cases} 2t & -1 < t < 1 \\ 0 \end{cases}$$

Point (a) $(-1,-2)$
Point (b) $(1,2)$

$$y-(-2) = \frac{2-(-2)}{1-(-1)}(x-(-1))$$

$$y+2 = \frac{4}{2}(x+1)$$

$$y+2 = 2x+2$$

$$y = 2x$$

$$x(t) = 2t$$

Since function is an odd function

$$a_n = 0,\ a_0 = \frac{1}{T}\int_{-1}^{1} 2tdt = \frac{1}{2}\times 0 = 0$$

$$b_n = \frac{2}{T}\int_{-1}^{1} t\sin(n\pi t)dt = \frac{2}{T}\left[\frac{-t\cos n\pi t}{n\pi}\Bigg|_{-1}^{1} + \frac{1}{n^2\pi^2}\cos n\pi t\Bigg|_{-1}^{1}\right]$$

2.3 CONVERGENCE OF FOURIER SERIES – DIRICHLET CONDITIONS

Existence of Fourier Series: The conditions under which a periodic signal can be represented by an F.S. are known as Dirichlet conditions. F.P. → Fundamental Period

(1) The function $x(t)$ has only a finite number of maxima and minima, if any within the F.P.
(2) The function $x(t)$ has only a finite number of discontinuities, if any within the F.P.
(3) The function $x(t)$ is absolutely integrable over one period, that is

$$\int_0^T |x(t)|dt < \infty$$

2.4 PROPERTIES OF CONTINUOUS FOURIER SERIES

(1) Linearity: If $x_1(t)$ and $x_2(t)$ are two periodic signals with period T with F.S.C. C_n and D_n then F.C. of linear combination of $x_1(t)$ and $x_2(t)$ are given by

$$\text{FS}[Ax_1(t)+Bx_2(t)] = AC_n + BD_n$$

Proof: If $z(t) = Ax_1(t) + Bx_2(t)$

$$a_n = \frac{1}{T}\int_{t_0}^{t_0+T} [Ax_1(t)+Bx_2(t)]\, e^{-jnw_0t} = \frac{A}{T}\int_T x_1(t)e^{-jnw_0t}dt + \frac{B}{T}\int_T x_2(t)e^{-jnw_0t}dt$$

$$a_n = AC_n + BD_n$$

(2) Time shifting: If the F.S.C. of $x(t)$ are C_n then the F.C. of the shifted signal $x(t-t_0)$ are

$$\text{FS}[x(t-t_0)] = e^{-jnw_0\ t_0}C_n$$

$$\text{Let } t - t_0 = \tau$$

$$dt = d\tau$$

$$B_n = \frac{1}{T}\int_T x(t-t_0)e^{-jnw_0t}dt$$

$$= \frac{1}{T}\int_T x(\tau)e^{-jnw_0(t_0+\tau)}d\tau = \frac{1}{T}\int_T x(\tau)e^{-jnw_0\tau}d\tau \cdot e^{-jnw_0t_0}$$

$$B_n = e^{-jnw_0\ t_0} \cdot C_n$$

(3) Time reversal: $\text{FS}[x(-t)] = C_{-n}$

$$B_n = \frac{1}{T}\int_T x(-t)e^{-jnw_0t}dt = \frac{1}{T}\int_T x(-t)e^{-j(-n)w_0T}dt$$

$$-t = \tau$$

$$dt = -d\tau$$

$$= \frac{1}{T}\int_{-T} x(\tau)e^{-j(-n)w_0\tau}d\tau = C_{-n}$$

Example 8: Compute the exponential series of the following signal.

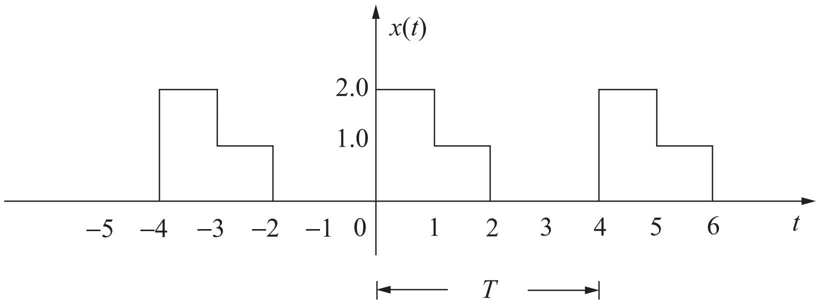


Fig. 2.8.

SOLUTION:

$$T = 4 \quad w_0 = \frac{\pi}{2}$$

$$C_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{1}{4}\left[\int_0^1 2dt + \int_1^2 dt\right] = \frac{3}{4}$$

$$C_n = \frac{1}{4}\left[\int_0^1 2e^{-jn\frac{\pi t}{2}}dt + \int_1^2 e^{-jn\frac{\pi t}{2}}dt\right]$$

$$= \frac{1}{4}\left\{\frac{-4}{jn\pi}\left[e^{-jn\frac{\pi}{2}} - 1\right] - \frac{2}{jn\pi}\left[e^{-jn\pi} - e^{-jn\frac{\pi}{2}}\right]\right\}$$

$$= \frac{-1}{2jn\pi}\left[2e^{\frac{-jn\pi}{2}} - 2 + e^{-jn\pi} - e^{-jn\frac{\pi}{2}}\right] = \frac{-1}{2jn\pi}\left[e^{-jn\frac{\pi}{2}} + e^{-jn\frac{\pi}{2}} - 2\right]$$

$$= -\frac{1}{jn\pi}\left[1 - \frac{1}{2}(-1)^n - \frac{1}{2}e^{-jn\frac{\pi}{2}}\right] \quad x(t) = \frac{3}{4} + \sum_{n=-\infty}^{\infty}\frac{1}{jn\pi}\left[e^{jn\frac{\pi}{2}} - \frac{1}{2}(-1)^n e^{jn\frac{\pi}{2}} - \frac{1}{2}\right]$$

Example 9:

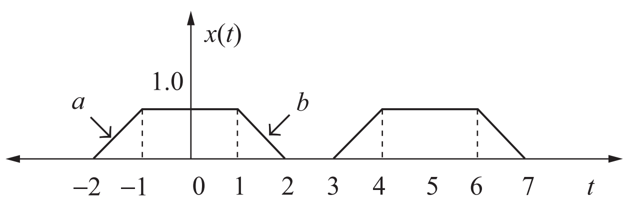


Fig. 2.9.

SOLUTION:

$$T = 5 \quad w_0 = \frac{2\pi}{5}$$

$$x(t) = \begin{cases} t+2 & -2 < t < -1 \\ 1.0 & -1 < t < 1 \\ 2-t & 1 < t < 2 \end{cases}$$

(a)

$$(-2,0)(-1,1)$$

$$(y-1) = \frac{-1}{-1}(x+1)$$

$$y = t+2$$

(b) $(1,1)(2,0)$

$$y - 0 = \frac{1}{-1}(x-2)$$

$$y = -x+2 = -t+2$$

$$C_0 = \frac{1}{5}\left[\int_{-2}^{-1}(t+2)dt + \int_{-1}^{1} dt + \int_{1}^{2}(2-t)dt\right]$$

$$C_0 = \frac{3}{5}$$

$$C_n = \frac{1}{5}\left[\underbrace{\int_{-2}^{-1}(t+2)e^{-j\frac{2n\pi}{5}}dt}_{A} + \underbrace{\int_{-1}^{1} e^{-j\frac{2n\pi}{5}}dt}_{B} + \underbrace{\int_{1}^{2}(2-t)e^{-j\frac{2n\pi}{5}}dt}_{C}\right]$$

$$A = \int_{-2}^{-1} e^{-j\frac{2n\pi}{5}t}dt + \int_{-2}^{-1} 2e^{-j\frac{2n\pi}{5}t}dt$$

$$A = -\frac{1}{j\phi}\left\{te^{-j\phi}\int_{-2}^{-1}\right\} + \frac{1}{\phi^2}e^{-j\phi}\int_{-2}^{-1} + \frac{2}{-j\phi}e^{j}\frac{2n\pi}{5}\int_{-2}^{-1}$$

$$= \frac{5}{j\,2n\pi}\left(-e^{j\frac{2n\pi}{5}} + 2e^{j\frac{4n\pi}{5}}\right) + \frac{25}{4n^2\pi^2}\left(e^{j\frac{2n\pi}{5}} - e^{j\frac{4n\pi}{5}}\right) - \frac{10}{2n\pi j}$$

$$A = \frac{5}{j2n\pi}\left(-e^{j\frac{2n\pi}{5}} + 4e^{j\frac{4n\pi}{5}}\right) + \frac{25}{4n^2\pi^2}\left(e^{j\frac{2n\pi}{5}} - e^{j\frac{4n\pi}{5}}\right)$$

$$B = \frac{e^{j\frac{2n\pi}{5}} - e^{-j\frac{2n\pi}{5}}}{j\frac{2n\pi}{5}} = \frac{5}{j\,2n\pi}\left(e^{j\frac{2n\pi}{5}} - e^{-j\frac{2n\pi}{5}}\right)$$

$$C = \frac{-10}{j\,2n\pi}\left(e^{-j\frac{4n\pi}{5}} - e^{-j\frac{2n\pi}{5}}\right) + \frac{10}{j\,2n\pi}e^{-j\frac{4n\pi}{5}} - \frac{5}{j\,2n\pi}e^{-j\frac{2n\pi}{5}} - \frac{25}{4n^2\pi^2}e^{-j\frac{4n\pi}{5}} + \frac{25}{4n^2\pi^2}e^{j\frac{n2\pi}{5}}$$

$$C_n = \frac{1}{5}\left[\frac{25}{n^2 4\pi^2}\left(e^{\frac{j2n\pi}{5}} - e^{\frac{j4n\pi}{5}}\right) - \frac{25}{4n^2\pi^2}\left(e^{-\frac{j4n\pi}{5}} - e^{-\frac{j2n\pi}{5}}\right)\right]$$

$$C_n = \frac{5}{2n^2\pi^2}\left[\cos\left(\frac{2\pi n}{5}\right) - \cos\left(\frac{4\pi n}{5}\right)\right]$$

Example 10: For the continuous-time periodic signal

$$x(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{5\pi}{3}t\right)$$

Determine the fundamental frequency w_0 and the Fourier series coefficients C_n such that

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jnw_0 t}$$

SOLUTION:

Given

$$x(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{5\pi}{3}t\right)$$

The time period of the signal $\cos\left(\frac{2\pi}{3}t\right)$ is

$$T_1 = \frac{2\pi}{w_1} = \frac{2\pi}{2\frac{\pi}{3}} = 3\,\text{sec}$$

The time period of the signal $\sin\left(5\frac{\pi}{2}t\right)$ is

$$T_2 = 2\frac{\pi}{w_2} = \frac{2\pi}{5\frac{\pi}{3}} = \frac{6}{5}\,\text{sec}$$

$\frac{T_1}{T_2} = \frac{3}{\frac{6}{5}} = \frac{5}{2}$ ratio of two integers, rational number, hence periodic.

$$2T_1 = 5T_2$$

The fundamental period of the signal $x(t)$ is

$$T = 2T_1 = 5T_2 = 6\,\text{sec}$$

and the fundamental frequency is

$$w_0 = \frac{2\pi}{T} = \frac{2\pi}{6} = \frac{\pi}{3}$$

$$x(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{5\pi}{3}t\right)$$

$$= 2 + \cos(2w_0 t) + 4\sin(5w_0 t)$$

$$= 2 + \frac{\left(e^{j2w_0 t} + e^{-j2w_0 t}\right)}{2} + \frac{4\left(e^{j5w_0 t} - e^{-j5w_0 t}\right)}{2j}$$

$$= 2 + 0.5\left(e^{j2w_0 t} + e^{-j2w_0 t}\right) - 2j\left(e^{j5w_0 t} - e^{-j5w_0 t}\right)$$

$$x(t) = 2je^{+j(-5)w_0 t} + 0.5e^{+j(-2)w_0 t} + 2 + 0.5e^{+j2w_0 t} - 2je^{+j5w_0 t}$$

$$a_n = \frac{2}{T}\int_{-\pi}^{\pi} x(t)\cos nt dt = \frac{4}{T}\int_{-\pi}^{0}\left(\frac{2t}{\pi}+1\right)\cos nt\ dt$$

$$= \frac{4}{2\pi}\left\{\frac{2t}{n\pi}\sin t + \frac{\sin nt}{n} - \int_{-\pi}^{0}\frac{2}{\pi}\sin nt\ dt\right\}$$

$$= \frac{1}{\pi}\left\{\frac{2t}{\pi}\sin nt + \sin nt + \frac{2}{n^2\pi}\cos nt\int_{-\pi}^{0}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{2}{n^2\pi} + \frac{2}{n^2\pi}\cos nt\right\} = \frac{4}{n^2\pi^2}\left\{1-\cos n\pi\right\} = \frac{4}{n^2\pi^2}\left(1-(-1)^n\right)$$

$$a_n = \begin{cases} 0 & n \text{ even } 2,\ 4,\ 6,\ 8,\cdots \\ \frac{8}{n^2\pi^2} & n \text{ odd } 1,\ 3,\ 5,\ 7,\cdots \end{cases}$$

2.5 FOURIER TRANSFORM

2.5.1 Definition

Let $x(t)$ be a signal which is a function of time t. The Fourier transform of $x(t)$ is given as

Fourier transform

$$\mathrm{X}\ (jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt \tag{1}$$

or

$$\mathrm{X}\ (if) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \tag{2}$$

Since $w = 2\pi f$

Similarly, $x(t)$ can be recovered from its Fourier transform $X(jw)$ by using Inverse Fourier transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{jwt}dw \tag{3}$$

$$x(t) = \int_{-\infty}^{\infty} X(if)e^{j2\pi ft}dt \tag{4}$$

Fourier transform $X(jw)$ is the complex function of frequency w. Therefore, it can be expressed in the complex exponential form as follows:

$$X(jw) = |X(jw)|e^{j\,\underline{|X(jw)}}$$

Here $|X(jw)|$ is the amplitude spectrum of $x(t)$ and $\underline{|X(jw)}$ is phase spectrum.

For a real-valued signal

(1) Amplitude spectrum is symmetric about vertical axis c (even function.)
(2) Phase spectrum is anti-symmetrical about vertical axis c (odd function.)

2.5.2 Existence of Fourier transform (Dirichlet's condition)

The following conditions should be satisfied by the signal to obtain its F.T.

(1) The function $x(t)$ should be single valued in any finite time interval T.

(2) The function $x(t)$ should have at the most finite number of discontinuities in any finite time interval T.

(3) The function $x(t)$ should have finite number of maxima and minima in any finite time interval T.

(4) The function $x(t)$ should be absolutely integrable, i.e.

$$\int_{-\infty}^{\infty} |x(t)|dt < \infty$$

- These conditions are sufficient, but not necessary for the signal to be Fourier transformable.
- A physically realizable signal is always Fourier transformable. Thus, physical realizability is the sufficient condition for the existence of F.T.
- All energy signals are Fourier transformable.

$$j\frac{d}{dw}X(jw) = FT(tx(t))$$
$$FT(tx(t)) = j\frac{d}{dw}X(jw)$$

Example 12: Obtain the F.T. of the signal $e^{-at}u(t)$ and plot its magnitude and phase spectrum.

SOLUTION:

$x(t) = e^{-at}u(t)$

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt = \int_{0}^{\infty} e^{-(a+j2\pi f)t}dt$$
$$X(f) = \frac{1}{a+j2\pi f}$$

To obtain the magnitude and phase spectrum:

$$|X(f)| = \frac{a-j2\pi f}{a^2+(2\pi f)^2} = \left(\frac{a}{a^2+4\pi^2 f^2}\right)A - j\left(\frac{2\pi f}{a^2+4\pi^2 f^2}\right)B$$
$$|X(f)| = \sqrt{A^2+B^2} = \frac{1}{\sqrt{a^2+4\pi^2 f^2}} = \frac{1}{\sqrt{a^2+w^2}}$$
$$|X(f)| = \tan^{-1}\left[\frac{-2\pi f}{a}\right] = -\tan^{-1}\left(\frac{w}{a}\right)$$
$$\text{for } a = 1,\ |X(f)| = \frac{1}{\sqrt{1+w^2}},\ \underline{|X(f)} = -\tan^{-1} w$$

w	0	1	2	3	4	5	10	15	25	8
$\lvert X(w)\rvert$	1	.707	0.447	0.316	0.242	0.196	0.09	0.066	0.03	0
$\underline{\lvert X(w)}$	0	45°	−63.4	−71.5	−75.9	−78.6	−84.2	−86.2	−87.7	−90°

(ii) $x(t) = e^{-a|t|} = \begin{cases} e^{-at} & t > 0 \\ e^{at} & t > 0 \end{cases}$

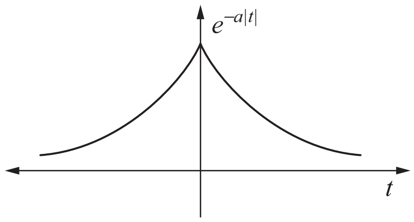


Fig. 2.13. Graphical representation of $e^{-a|t|}$

$$x(w) = \frac{1}{a+jw} + \frac{1}{a-jw} = \frac{2a}{a^2+w^2}$$

$$\text{for } a = 1 \; X(w) = \frac{2}{1+w^2}$$

$$|X(w)| = \frac{2}{1+w^2} \quad \frac{|X(w)}{} = 0$$

w (in radians)	$-\infty$	-10	-5	-3	-2	-1	0	1	2	3	4	5	10	∞
$\lvert X(w) \rvert$	0	0.019	0.0769	0.2	0.4	1	2	1	0.4	0.2	.1176	0.0769	0.019	0

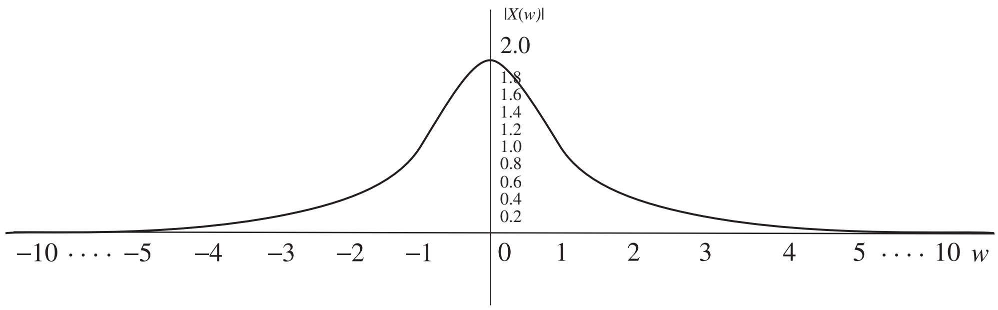


Fig. 2.14. Magnitude plot

(iii) $x(t) = e^{-a|t|}\ \text{sgn}(t)$

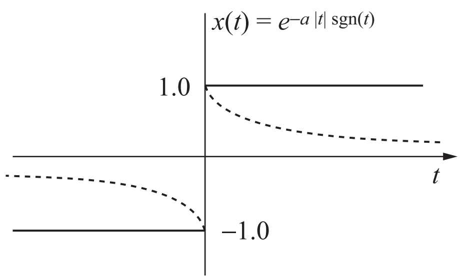


Fig. 2.15. Graphical representation of $e^{-a|t|}\text{sgn}(t)$

(ii)
$$x(t) = 1$$
$$X(w) = \int_{-\infty}^{\infty} e^{-jwt} dt = \infty$$

This means Dirichlet condition is not satisfied. But its F.T. can be calculated with the help of duality property.

$$\delta(t) \xleftrightarrow{FT} 1$$

Duality property states that: $x(t) \xleftrightarrow{FT} X(w)$ then

$$X(t) \xleftrightarrow{FT} 2\pi x(-w)$$

Here $X(t) = 1$, then $x(-w)$ will be

$$x(t) = \delta(t); \quad X(w) = 1$$

then $X(t) = 1; \quad 1 \xleftrightarrow{FT} 2\pi\delta(-w)$

We know that $\delta(w)$ will be an even function of w, since it is impulse function.

Hence, $\delta(-w) = \delta(w)$. Then above equation becomes

$$1 \xleftrightarrow{FT} 2\pi\delta(-w)$$

Thus, if $x(t) = 1$, then $X(w) = 2\pi\delta(w)$

(iii) $x(t) = \text{sgn}(t) \qquad \text{sgn}(t) = \begin{Bmatrix} 1 & t > 0 \\ -1 & t < 0 \end{Bmatrix}$

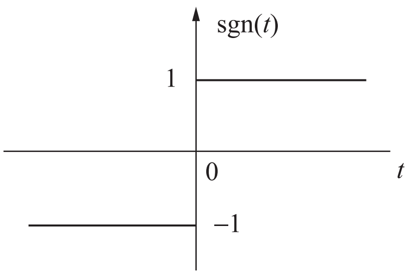


Fig. 2.17. Graphical representation of sgn(t)

$$x(t) = 2u(t) - 1$$

Differentiating both the sides

$$\frac{d}{dt}x(t) = 2\frac{d}{dt}u(t) = 2\delta(t)$$

Taking the F.T. of both sides

$$F\left[\frac{d}{dt}x(t)\right] = 2F[\delta(t)]$$
$$jwX(w) = 2$$
$$X(w) = \frac{2}{jw}$$
$$X(w) = \int_{0}^{\infty} e^{-jwt} dt - \int_{-\infty}^{0} e^{-jwt} dt$$

(iv)
$$x(t) = u(t)$$
$$\text{sgn}(t) = 2u(t) - 1$$
$$2u(t) = 1 + \text{sgn}(t)$$

Taking F.T. of both sides

$$2F[2u(t)] = F(1) + F[\text{sgn}(t)] = 2\pi\delta(w) + \frac{2}{jw}$$
$$2u(t) \xleftrightarrow{FT} 2\pi\delta(w) + \frac{2}{jw}$$
$$u(t) \xleftrightarrow{FT} \pi\delta(w) + \frac{1}{jw}$$

Properties of unit impulse:

(1) $\int_{-\infty}^{\infty} x(t)\delta(t) = x(0)$

(2) $x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$

(3) $\int_{-\infty}^{\infty} x(t)\delta(t - t_0)dt = x(t_0)$

(4) $\delta(at) = \frac{1}{|a|}\delta(t)$

(5) $\int_{-\infty}^{\infty} x(\tau)\delta(t - x)dt = x(t)$

(6) $\delta(t) = \frac{d}{dt}u(t)$

Example 15: Obtain the F.T. of a rectangular pulse shown in Fig. 2.18.

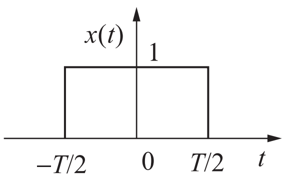


Fig. 2.18. Rectangular pulse

SOLUTION:

$$X(w) = \int_{\frac{-T}{2}}^{\frac{T}{2}} e^{-jwt}dt = \frac{-1}{jw}\left[e^{-jw\frac{T}{2}} - e^{jw\frac{T}{2}}\right] = \frac{2}{w}\sin\left(\frac{wT}{2}\right)$$
$$X(w) = T\frac{\sin\left(\pi\frac{wT}{2\pi}\right)}{\pi\frac{wT}{2\pi}} = \sin c\left(\frac{wT}{2\pi}\right) = T\frac{\sin\left(\pi\frac{wT}{2\pi}\right)}{\pi\frac{wT}{2\pi}}$$

Sampling function or interpolating function or filtering function denoted by $S_a(x)$ or $\sin c(x)$ as shown in figure.

$$\sin c(x) = \frac{\sin \pi x}{\pi x}$$

(1) $\sin c(x) = 0$ when $x = \pm n\pi$
(2) $\sin c(x) = 1$ when $x = 0$ (using L'Hospital's rule)
(3) $\sin c(x)$ is the product of an oscillating signal $\sin x$ of period 2π and a decreasing signal $\frac{1}{x}$. Therefore, $\sin c(x)$ is making sinusoidal of oscillations of period 2π with amplified decreasing continuously as $\frac{1}{x}$.

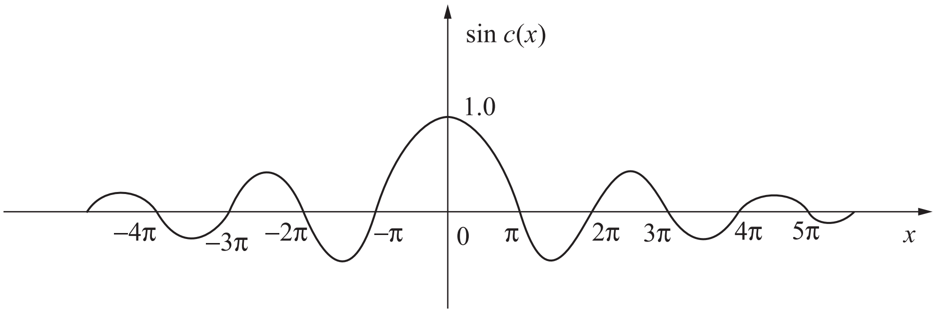


Fig. 2.19. Sine function

$$\sin cx = \frac{\sin \pi x}{\pi x}; \qquad \sin c(0) = \frac{0}{0} = 1 \text{ L'Hospital rule}$$

$$\sin c(1) = \frac{\sin \pi}{\pi} = 0; \quad \sin c(-1) = 0$$

$$\sin c(2) = 0; \qquad \sin c(-2) = 0$$

$\sin c(1/4) = 0.9$ $\qquad$ $\sin c(-1/4) = 0.9$
$\sin c(2/4) = .6366$ $\qquad$ $\sin c(-0.5) = .6366$
$\sin c(3/4) = 0.3$ $\qquad$ $\sin c(-7.5) = .3$
$\sin c(1.5) = -.2122$ $\qquad$ $\sin c(-1.5) = -.2122$
$\sin c(2.5) = .1273$ $\qquad$ $\sin c(2.5) = .1273$

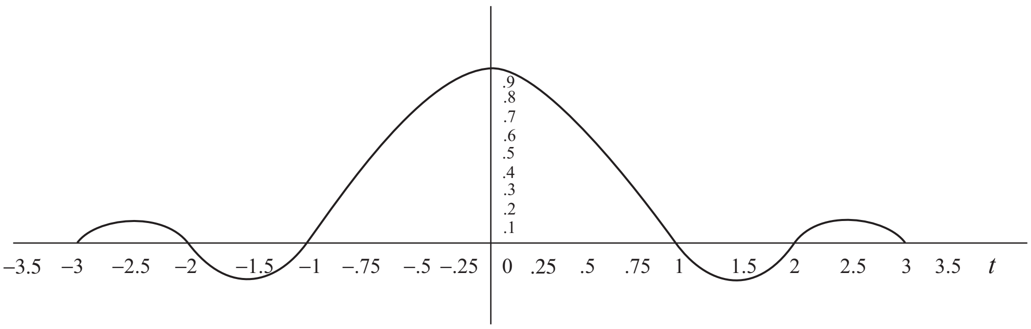


Fig. 2.20. Sine function

Example 17: Obtain F.T. and spectrums of following signals:

(i) $x(t) = \cos w_0 t$ (ii) $x(t) = \sin w_0 t$

SOLUTION:

(i)
$$x(t) = \cos w_0 t = \frac{1}{2}e^{jw_0 t} + \frac{1}{2}e^{-jw_0 t}$$

$$1 \xleftrightarrow{FT} 2\pi\delta(w);\ \frac{1}{2} \xleftrightarrow{FT} \pi\delta(w)$$

Frequency shifting property states that $e^{j\beta t}x(t) \xleftrightarrow{FT} X(w-\beta)$

$$\frac{1}{2}e^{jw_0 t} \xleftrightarrow{FT} \pi\delta(w-w_0)$$

$$\frac{1}{2}e^{-jw_0 t} \xleftrightarrow{FT} \pi\delta(w+w_0)$$

$$F[x(t)] = FT\left\{\frac{1}{2}e^{jw_0 t} + \frac{1}{2}e^{-jw_0 t}\right\}$$

$$X(w) = \pi[\delta(w-w_0) + \delta(w+w_0)]$$

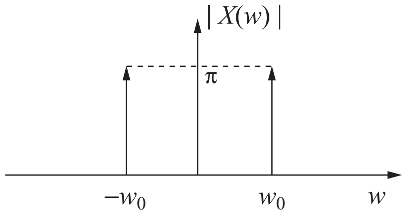


Fig. 2.22. Magnitude plot of $\cos w_0 t$

(ii)
$$x(t) = \sin w_0 t$$

$$X(w) = \frac{\pi}{j}[\delta(w-w_0) - \delta(w+w_0)]$$

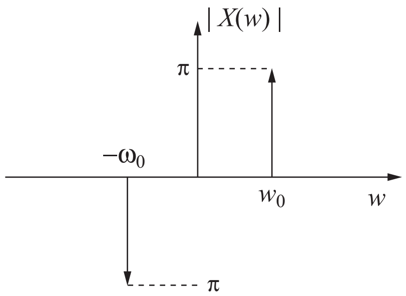


Fig. 2.23. Magnitude plot of $\sin w_0 t$

Example 18: Obtain the F.T. of

$$x(t) = te^{-at}u(t)$$

from property of Fourier transform $\text{FT}[tx(t)] = j\frac{d}{dw}X(w)$

$$\text{FT}\left[e^{-at}\right] = \frac{1}{a+jw}$$

$$\text{FT}(te^{-at}) = j\frac{d}{dw}\left(\frac{1}{a+jw}\right) = j\frac{(a+jw)\frac{d}{dw}(1) - 1\frac{d}{dw}(a+jw)}{(a+jw)^2} = \frac{1}{(a+jw)^2}$$

Inverse Fourier Transform: (IFT)

Example 19: Find the IFT of

(i) $X(w) = \frac{2jw+1}{(jw+2)^2}$ by partial fraction expansions

(ii) $X(w) = \frac{1}{(a+jw)^2}$ by convolution property

(iii) $X(w) = e^{-|w|}$

(iv) $X(w) = e^{-2w}u(w)$

SOLUTION:

(i)

$$X(w) = \frac{A}{jw+2} + \frac{B}{(jw+2)^2};\ 2jw+1 = A(jw+2)+B \quad A=2 \quad 2A+B=1 \quad B=-3$$

$$X(w) = \frac{2}{jw+2} - \frac{3}{(jw+2)^2}$$

$$x(t) = 2e^{-2t}u(t) - 3te^{-2t}u(t)$$

(ii)

$$X(w) = \frac{1}{(a+jw)^2} = \frac{1}{(a+jw)(a+jw)} = X_1(w)X_2(w)$$

$$X_1(w) = \frac{1}{a+jw},\ X_2(w) = \frac{1}{a+jw}$$

$$x_1(t) = e^{-at}u(t),\ x_2(t) = e^{-at}u(t)$$

Using convolution property

$$x(t) = x_1(t)^* x_2(t)$$

$$x(t) \overset{\text{FT}}{\longleftrightarrow} X(w)$$

$$x_1(t)^* x_2(t) \overset{\text{FT}}{\longleftrightarrow} X_1(w)X_2(w)$$

$$x(t) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-a(t-\tau)}u(t-\tau)d\tau \qquad \begin{cases} u(\tau) = 1\ \tau \le 0 \\ u(t-\tau) = 1 \quad t \le \tau \end{cases}$$

$$= \int_0^t e^{-at}d\tau = te^{-at}u(t)$$

Example 20: Find the F.T. of the function

$$x(t-t_0) = e^{-(t-t_0)}u(t-t_0)$$

SOLUTION:

$$\text{If } F[x(t)] = X(w)$$
$$\text{then FT}[x(t-t_0)] = e^{-jwt_0}X(w)$$
$$F\left[e^{-t}u(t)\right] = \frac{1}{1+jw}$$
$$F\left[e^{-(t-t_0)}u(t-t_0)\right] = \frac{e^{-jwt_0}}{1+jw}$$

Example 21: Find the F.T. of the function

$$x(t) = [u(t+1) - u(t-1)]\cos 2\pi t$$

SOLUTION:

$$\text{FT}(\cos 2\pi t) = \text{FT}\left(\frac{e^{j2\pi t} + e^{-j2\pi t}}{2}\right)$$
$$\text{FT}[1] = 2\pi\delta(w)$$
$$\text{FT}[e^{jw_0t}] = 2\pi\delta(w-w_0)$$
$$F[\cos 2\pi t] = \pi\delta(1w-2\pi) + \pi\delta(w+2\pi) \quad (1)$$
$$F[u(t+1)-u(t-1)] = \int_{-1}^{1} e^{-jwt}dt = -\frac{1}{jw}\left(e^{-jw} - e^{jw}\right) = \frac{2\sin w}{w} \quad (2)$$
$$F[x(t)] = F[\{u(t+1) - u(t-1)\}\cos 2\pi t]$$

$x(t)$ is multiplication of (1) and (2), so by using multiplication property

$$x(t)y(t) \overset{\text{FT}}{\longleftrightarrow} \frac{1}{2\pi}X_1(w)^*Y_1(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\tau)Y(w-\tau)d\tau$$

$$X(w) = \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} \frac{2\sin\tau}{\tau}\pi\delta(w-2\pi-\tau) + \delta(w+2\pi-\tau)\right]d\tau$$

$$X(w) = \int_{-\infty}^{\infty} \frac{\sin\tau}{\tau}\delta(w-2\pi-\tau)d\tau + \int_{-\infty}^{\infty} \frac{\sin\tau}{\tau}\delta(w+2\pi-\tau)d\tau$$

Since $\displaystyle\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)$

$$X(w) = \sin(w-2\pi)/(w-2\pi) + \sin(w+2\pi)/(w+2\pi)$$

Example 22: Determine the Fourier transform of a triangular function as shown in figure.

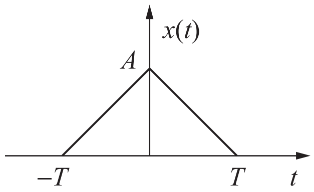
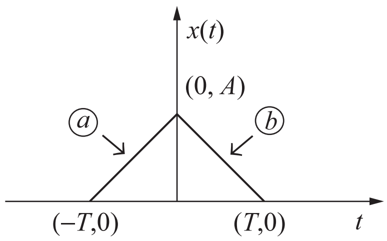


Fig. 2.24. Triangular pulse

SOLUTION:



Equation of line (a) is

$$x(t) = A\left(\frac{t}{T} + 1\right)$$

Equation of line (b) is

$$x(t) = A\left(1 - \frac{t}{T}\right)$$

Mathematically, we can write $x(t)$ as

$$x(t) = A\left(\frac{t}{T} + 1\right)[u(t+T) - u(t)] + A\left(1 - \frac{t}{T}\right)[u(t) - u(t-T)]$$

$$x(t) = \frac{A}{T}(t+T)[u(t+T) - u(t)] + \frac{A}{T}(T-t)[u(t) - u(t-T)]$$

$$x(t) = \frac{A}{T}\left\{(t+T)u(t+T) - (t+T)u(t)\right\} + \frac{A}{T}\left\{[(T-t)u(t) - (T-t)u(t-T)]\right\}$$

$$x(t) = \frac{A}{T}\left\{r(t+T) - tu(t) - Tu(t)\right\} + \frac{A}{T}\left\{Tu(t) - tu(t) + r(t-T)\right\}$$

$$= \frac{A}{T}\left\{r(t+T) - r(t) - Tu(t)\right\} + \frac{A}{T}\left\{Tu(t) - r(t) + r(t-T)\right\}$$

$$= \frac{A}{T}\left[\left\{r(t+T) - 2r(t) + r(t-T)\right\}\right]$$

$$X(jw) = \frac{A}{T}\left[\frac{e^{jwT}}{(jw)^2} - \frac{2}{(jw)^2} + \frac{e^{-jwT}}{(jw)^2}\right]$$

$$\Pi(t) = \text{rect}(t) = \begin{cases} 1 & -\frac{1}{2} < t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{rect}(t-5) = \begin{cases} 1 & -\frac{1}{2} \leq t-5 < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{rect}(t-5) = \begin{cases} 1 & \frac{9}{2} \leq t \leq \frac{11}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt = \int_{-\infty}^{\infty} rect(t-5)e^{-jwt}dt$$

$$= \int_{9/2}^{11/2} e^{-jwt}dt = \left.\frac{e^{-jwt}}{-jw}\right|_{9/2}^{11/2}$$

$$= \frac{e^{-\frac{j11w}{2}} - e^{-\frac{9jw}{2}}}{-jw} = \frac{e^{-9j\frac{w}{2}} - e^{-11j\frac{w}{2}}}{jw}$$

$$= \frac{e^{-5jw}e^{jw/2} - e^{-5jw}e^{-jw/2}}{jw} = \frac{2e^{-5jw}\left(e^{jw/2} - e^{-jw/2}\right)}{w2j}$$

$$= \frac{2e^{-5jw}}{w}\sin\frac{w}{2} = e^{-5jw}\left(\frac{\sin\frac{w}{2}}{\frac{w}{2}}\right)$$

$$X(jw) = e^{-5jw}S_a\left(\frac{w}{2}\right)$$

2.6 PROPERTIES OF CONTINUOUS-TIME FOURIER TRANSFORM

(1) Linearity

If FT $(x_1(t)) = X_1(jw)$
and FT $(x_2(t)) = X_2(jw)$
Then linearity property states that

$$\text{FT}(Ax_1(t) + Bx_2(t)) = AX_1(jw) + BX_2(jw)$$

where A and B are constants.

Proof:

$$\text{Let } r(t) = Ax_1(t) + Bx_2(t)$$

$$\text{FT}(r(t)) = R(jw) = \int_{-\infty}^{\infty} r(t)e^{-jwt}dt$$

$$= \int_{-\infty}^{\infty} (Ax_1(t) + Bx_2(t))\, e^{-jwt}dt$$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-j(-w)\tau} d\tau$$

$$F(x(t)) = X(-jw)$$

(4) Time shifting

If FT $(x(t)) = X(jw)$
then FT $(x(t-t_0)) = e^{-jwt_0} X(jw)$

Proof:

$$\text{Let } r(t) = x(t-t_0)$$

$$R(jw) = \int_{-\infty}^{\infty} r(t) e^{-jwt} dt = \int_{-\infty}^{\infty} x(t-t_0) e^{-jwt} dt$$

$$R(jw) = \text{FT}(x(t-t_0)) = \int_{-\infty}^{\infty} x(t-t_0) e^{-jwt} dt$$

Let $t - t_0 = \tau$ $\qquad dt = d\tau$

$$\text{FT } (x(t-t_0)) = \int_{-\infty}^{\infty} x(\tau) e^{-jw(t_0+\tau)} d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-jwt} e^{-jwt_0} d\tau$$

$$= e^{-jwt_0} \int_{-\infty}^{\infty} x(\tau) e^{-jw\tau} d\tau$$

FT $(x(t-t_0)) = e^{-jwt_0} X(jw)$. Similarly, FT $(x(t+t_0)) = e^{jwt_0} X(jw)$

So FT $(x(t \pm t_0)) = e^{\pm jwt_0} X(jw)$

(5) Frequency shifting

$$\text{If FT } (x(t)) = X(jw)$$

$$\text{FT } (e^{jw_0 t} x(t)) = X(j(w-w_0))$$

$$\text{Let } r(t) = e^{jw_0 t} x(t)$$

$$\text{FT } (r(t)) = \text{FT} \left(e^{jw_0 t} x(t)\right) = R(jw) = \int_{-\infty}^{\infty} e^{jw_0 t} x(t) e^{-jwt} dt$$

$$\text{FT} \left(e^{jw_0 t} x(t)\right) = \int_{-\infty}^{\infty} x(t) e^{-j(w-w_0)t} dt$$

$$\text{Let } w - w_0 = w'$$

$$= \int_{-\infty}^{\infty} x(t)e^{-jw't}dt$$

$$\text{FT}\left(e^{jw_0t}x(t)\right) = X(jw') = X(j(w - w_0))$$

$$\text{Similarly, FT}(e^{-jw_0t}x(t)) = X(j(w + w_0))$$

$$\text{We can write as FT}\left(e^{\pm jw_0t}x(t)\right) = X(j(w \mp w_0))$$

(6) Duality or symmetry property

If FT $(x(t)) = X(jw)$

then FT $(x(t)) = 2\pi x(-jw)$

Proof:

We know that $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{jwt}dw$

Replacing t by $-t$, we get

$$x(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{-jwt}dw$$

$$2\pi\, x(-t) = \frac{2\pi}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{-jwt}dw$$

$$2\pi\, x(-t) = \int_{-\infty}^{\infty} X(jw)e^{-jwt}dw$$

Interchanging t by jw

$$2\pi\, x(-jw) = \int_{-\infty}^{\infty} X(t)e^{-jwt}dt$$

$$2\pi\, x(-jw) = \text{FT}(X(t))$$

(7) Convolution in time domain

If FT $(x_1(t)) = X_1(jw)$ and FT $(x_2(t)) = X_2(jw)$

then FT $(x_1(t)^* x_2(t)) = X_1(jw)X_2(jw)$

i.e., convolution in time domain becomes multiplication in frequency domain.

Proof:

$$r(t) = x_1(t)^* x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau$$

$$\text{FT}(r(t)) = R(jw) = \int_{-\infty}^{\infty} r(t)e^{-jwt}dt$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau \right) e^{-jwt}dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau\, e^{-jwt}dt$$

$$= \int_{-\infty}^{\infty} x_1(\tau)d\tau \int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-jwt}dt$$

Let $t - \tau = \propto$ so $dt = d\propto$

$$\text{FT}\left[x_1(t)^* x_2(t)\right] = \int_{-\infty}^{\infty} x_1(t)d\tau \int_{-\infty}^{\infty} x_2(\propto)\, e^{-jw(\propto+\tau)}d\propto$$

$$= \int_{-\infty}^{\infty} x_1(\tau)d\tau \int_{-\infty}^{\infty} x_2(\propto)\, e^{-jw\propto}\, e^{-jw\tau}d\propto$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, e^{-jw\tau}d\tau \int_{-\infty}^{\infty} x_2(\propto)\, e^{-jw\propto}d\propto$$

$$\text{FT}\left[x_1(t)^* x_2(t)\right] = X_1(jw)\, X_2(jw)$$

(8a) Integration in time domain

If FT $(x(t)) = X(jw)$
then FT $\left(\int_{-\infty}^{t} x(\tau)d\tau\right) = \frac{1}{jw} \times (jw)$

Proof: Let $r(t) = \int_{-\infty}^{t} x(\tau)d\tau$
Differentiating w.r.t. t

$$\frac{dr(t)}{dt} = x(t) \Rightarrow \text{FT}(x(t)) = \text{FT}\left(\frac{d}{dt}r(t)\right)$$

From differentiation in time domain

$$X(jw) = jwX(jw)$$

$$R(jw) = \frac{1}{jw}X(jw)$$

$$\text{FT}(r(t)) = \text{FT}\left(\int_{-\infty}^{t} x(\tau)d\tau\right) = \frac{1}{jw}X(jw)$$

(8b) Differentiation in time domain

If FT $(x(t)) = X(jw)$

then $\left(\frac{d}{dt}x(t)\right) = jw \times (jw)$

Proof: We know that $x(t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} X(jw)e^{jwt}dw$. Differentiating both sides w.r.t. t

$$\frac{d}{dt}x(t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} X(jw)\left(\frac{d}{dt}e^{jwt}\right)dw$$

$$= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} jwX(jw)e^{jwt}dw$$

$$= j\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} (wX(jw))e^{jwt}dw$$

$$\frac{d}{dt}x(t) = j\ \mathrm{FT}^{-1}(wX(jw))$$

yields FT $\left(\frac{d}{dt}x(t)\right) = jwX(jw)$. On generalizing we get FT $\left(\frac{d^n}{dt^n}x(t)\right) = (jw)^n X(jw)$

(9) Differentiation in frequency domain

If FT $(x(t)) = X(jw)$

then FT $(tx(t)) = j\frac{d}{dw}X(jw)$

Proof: We know that $X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$

On differentiating both sides w.r.t. w

$$\frac{d}{dw}X(jw) = \int\limits_{-\infty}^{\infty} x(t)\left(\frac{d}{dw}e^{-jwt}\right)dt = -\int\limits_{-\infty}^{\infty} j\ t\ x(t)e^{-jwt}dt$$

Multiplying both sides by j

$$j\frac{d}{dw}X(jw) = \int\limits_{-\infty}^{\infty} (tx(t))e^{-jwt}dt \quad \text{since } j^2 = -1 \text{ or } -j^2 = 1$$

$$j\frac{d}{dw}X(jw) = \mathrm{FT}[t\ x(t)]$$

$$\mathrm{FT}[t\ x(t)] = j\ \frac{d}{dw}X(jw)$$

(10) Convolution in frequency domain (multiplication in time domain (multiplication theorem))

$$\text{If } \mathrm{FT}(x_1(t)) = X_1(jw) \text{ and } \mathrm{FT}[x_2(t)] = X_2(jw)$$

$$\mathrm{FT}(x_1(t)x_2(t)) = \frac{1}{2\pi}(X_1(jw)^* X_2(jw))$$

Proof:

$$E = \int_{-\infty}^{\infty} \left|x(t)^2\right| dt = \int_{-\infty}^{\infty} \left|x(t)x^*(t)dt \qquad (1)$$

$$\text{We know that } x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{+jwt}dw$$

$$\text{So } x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{-jwt}dw \qquad (2)$$

on putting (1)

$$= \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(jw)e^{-jwt}dw\right]dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(jw)\int_{-\infty}^{\infty} x(t)e^{-jwt}dt\, dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)X^*(jw)dw$$

$$= \int_{-\infty}^{\infty} \left|x(t)^2\right|dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|X(jw)\right|^2 dw$$

Relation between Laplace Transform and Fourier Transform

Fourier transform $X(jw)$ of a signal $x(t)$ is given as

$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt \qquad (1)$$

F.T. can be calculated only if $x(t)$ is absolutely integrable

$$= \int_{-\infty}^{\infty} \left|x(t)\right|dt < \infty \qquad (2)$$

Laplace transform $X(s)$ of a signal $x(t)$ is given as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \qquad (3)$$

We know that $s = \sigma + jw$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+jw)t}dt$$

$$X(s) = \int_{-\infty}^{\infty} \left[x(t)e^{-\sigma t}\right]e^{-jwt}dt \qquad (4)$$

Comparing (1) and (4), we find that L.T. of $x(t)$ is basically the F.T. of $[x(t)e^{-\sigma t}]$.

If $s = jw$, i.e. $\sigma = 0$, then eqn. (4) becomes $X(s) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt = X(jw)$

Thus, $X(s) = X(jw)$ when $\sigma = 0$ or $s = jw$

This means L.T. is same as F.T. when $s = jw$. The above equation shows that F.T. is special case of L.T. Thus, L.T. provides broader characterization compared to F.T., $s = jw$ indicates imaginary axis in complex s-plane.

2.7 APPLICATIONS OF FOURIER TRANSFORM OF NETWORK ANALYSIS

Example 24: Determine the voltage $V_{out}(t)$ to a current source excitation $i(t) = e^{-t}u(t)$ for the circuit shown in figure.

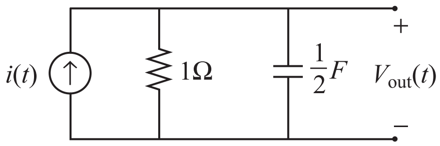
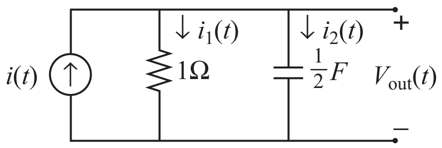


Fig. 2.26.

SOLUTION:



$$i(t) = i_1(t) + i_2(t)$$

$$i(t) = \frac{V_{out}(t)}{1} + \frac{1}{2}\frac{dV_{out}(t)}{dt}$$

$$\left\{ \begin{array}{l} \text{since } i = \frac{V}{R} \\ \text{and } i = c\frac{dv}{dt} \text{ or } v = \frac{1}{c}\int i\,dt \end{array} \right.$$

$$e^{-t}u(t) = V_{out}(t) + \frac{1}{2}\frac{dV_{out}(t)}{dt} \qquad (1)$$

On taking the z-transform on both sides

$$\frac{1}{1+jw} = V_{out}(jw)\left\{1 + \frac{jw}{2}\right\} = \frac{(2+jw)}{2}V_{out}(jw)$$

$$V_{out}(jw) = \frac{2}{(1+jw)(2+jw)} = \frac{A}{1+jw} + \frac{B}{2+jw}$$

$$V_{out}(jw) = \frac{2}{1+jw} - \frac{2}{2+jw}$$

$$\left\{ \begin{array}{l} A(2+jw) + B(1+jw) = 2 \\ 2A + B = 2 \\ A + B = 0 \; s_0 \; A = -B \\ 2A - A = 2; \quad A = 2, B = -2 \end{array} \right.$$

$$V_0(jw) = \frac{2}{6(jw)^2+7(jw)+1} = \frac{2}{(6jw+1)(jw+1)}$$

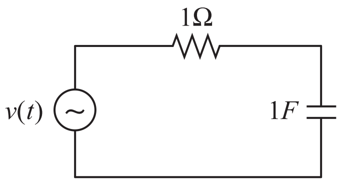
$$V_0(jw) = \frac{1/3}{(jw+1/6)(jw+1)} = \frac{A}{\frac{1}{6}+jw} + \frac{B}{1+jw}$$

$$V_0(jw) = \frac{2}{5\left(\frac{1}{6}+jw\right)} - \frac{2}{5(1+jw)} \tag{5}$$

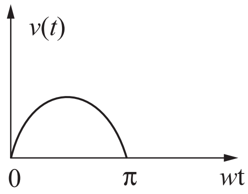
Taking inverse Fourier transform, we get

$$V_0(t) = \frac{2}{5}\left(e^{-t/6} - e^{-t}\right)u(t) \tag{6}$$

Example 26: Determine the response of current in the network shown in Fig. 2.28(a) when a voltage having the waveform shown in Fig. 2.28(b) is applied to it by using the Fourier transform.



(a)



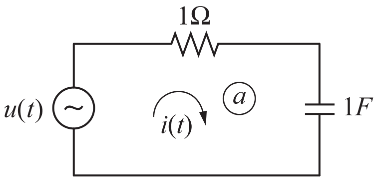
(b)

Fig. 2.28.

SOLUTION:

Waveform $V(t)$ is defined as

$$V(t) = \sin t(u(t) - u(t-\pi)) \tag{1}$$



Let $i(t)$ be the current in the loop. Applying KVL in loop

$$V(t) = 1\cdot i(t) + \frac{1}{1}\int_0^t i(t)dt = i(t) + \int_0^t i(t)dt \tag{2}$$

On taking Fourier transform of

$$V(jw) = \frac{1}{(jw)^2+1} + \frac{e^{-j\pi w}}{(jw)^2+1}$$

Since
$$\text{FT}\left[\sin t u(t)\right] = \frac{1}{(jw)^2+1}$$

$$\text{FT}\left[\sin t u(t-\pi)\right] = \frac{e^{-j\pi w}}{(jw)^2+1}$$

Solve using F.T. formula

$$V(jw) = \frac{1+e^{-j\pi w}}{(jw)^2+1} \tag{3}$$

$$V(jw) = I(jw) + \frac{1}{jw} I(jw)$$

$$V(jw) = \left(1 + \frac{1}{jw}\right) I(jw) = \frac{jw+1}{jw} I(jw)$$

$$I(jw) = \frac{jw}{jw+1} V(jw) \tag{4}$$

$$I(jw) = \frac{jw}{jw+1} \cdot \frac{(1+e^{-j\pi w})}{((jw)^2+1)} \quad \text{From (3)}$$

$$I(jw) = \frac{jw}{jw+1} \cdot \left\{ \frac{1}{(jw)^2+1} + \frac{e^{-j\pi w}}{(jw)^2+1} \right\}$$

$$= \underbrace{\frac{jw}{(jw+1)} \cdot \frac{1}{((jw)^2+1)}}_{I_1(jw)} + \underbrace{\frac{jw}{(jw+1)} \cdot \frac{1}{((jw)^2+1)} \cdot e^{-j\pi w}}_{I_2(jw)}$$

$$I_1(jw) = \frac{A}{jw+1} + \frac{Bjw + c}{((jw)^2+1)}$$

$$= \frac{-1/2}{(jw+1)} + \frac{\frac{1}{2}(jw+1)}{((jw)^2+1)}$$

$$i_1(t) = -\frac{1}{2} e^{-t} u(t) + \frac{1}{2} \cos tut + \frac{1}{2} \sin t\delta t + \frac{1}{2} \sin tu(t)$$

Since IFT $\left\{ \frac{1}{(jw)^2+1} \right\} = \sin tu(t)$

so IFT $\left(\frac{jw}{(jw)^2+1} \right) = \frac{d}{dt} \sin tu(t)$

Using differential in time domain property

$$\text{IFT}\left[\frac{jw}{(jw)^2+1} \right] = \cos tu(t) + \sin t\delta(t)$$

$$I_2(jw) = \frac{jw}{(jw+1)} \cdot \frac{1}{((jw)^2+1)} \cdot e^{-j\pi w}$$

$$I_2(jw) = I_3(jw) \cdot e^{-j\pi w}$$

Since $$I_3 = I_1(jw)$$

so $$i_3(t) = -\frac{1}{2} e^{-t} u(t) + \frac{1}{2} \cos tu(t) + \frac{1}{2} \sin t\delta(t) + \frac{1}{2} \sin tu(t)$$

From time shifting property $\text{FT}(x(t \pm t_0)) = e^{\pm jwt_0} \times (jw)$

$$i_2(t) = i_3(t-\pi)$$

$$= -\frac{1}{2}e^{-(t-\pi)}u(t-\pi) + \frac{1}{2}\cos(t-\pi)u(t-\pi) + \frac{1}{2}\sin(t-\pi)\delta(t-\pi) + \frac{1}{2}\sin(t-\pi)u(t-\pi)$$

so $$i(t) = \frac{1}{2} - \left[-e^{-t} + \cos t + \sin t\right]u(t) + \frac{1}{2}\sin t\,\delta(t) + \frac{1}{2}\left[-e^{-(t-\pi)} + \cos(t-\pi) + \sin(t-\pi)\right]u(t-\pi) + \frac{1}{2}\sin(t-\pi)\delta(t-\pi)$$

Example 27: For the *RC* circuit shown in figure.

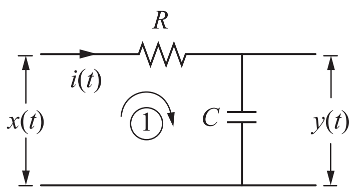


Fig. 2.29.

(a) Determine frequency response of the circuit.

(b) Find impulse response.

(c) Plot the magnitude and phase response for $RC = 1$.

SOLUTION:

Applying KVL in loop (1)

$$x(t) - Ri(t) - \frac{1}{C}\int_{-\infty}^{t} i(t)dt = 0$$

$$x(t) = Ri(t) + \frac{1}{C}\int_{-\infty}^{t} i(t)dt \qquad (1)$$

$$\begin{cases} Since \\ V_R = iR \\ V_c = \frac{1}{C}\int i(t)dt \end{cases}$$

and $$y(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)dt \qquad (2)$$

$$A = B - C$$

$$\left|H(jw)\right| = \frac{1}{\sqrt{1+w^2}} \tag{8}$$

$$H(jw) = 1 - (1 + jw)$$

$$= \tan^{-1}\frac{0}{1} - \tan^{-1} w = -\tan^{-1} w \tag{9}$$

For different values of w, we find $\left|H(jw)\right|$ and $H(jw)$

S. No	w	$\lvert H(jw)\rvert$	$H(jw)$
1–	$-\infty$	0	90°
2–	−50	0.0199	88.9°
3–	−20	0.0499	87.1°
4–	−10	0.099	84.3°
5–	−5	0.196	78.7°
6–	−2	0.447	63.4°
7–	−1	0.707	45°
8–	0	1	0
9–	1	0.707	−45°
10–	2	0.447	−63.4°
11–	5	0.196	−78.7°
12–	10	0.099	−84.3°
13–	20	0.0499	−87.1°
14–	50	0.0199	−88.9°
15–	∞	0	−90°

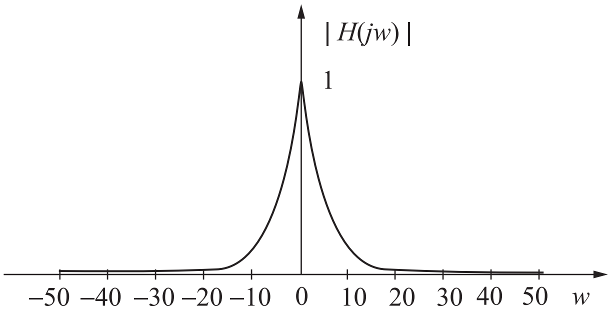


Fig. 2.30. Magnitude plot frequency response of the circuit

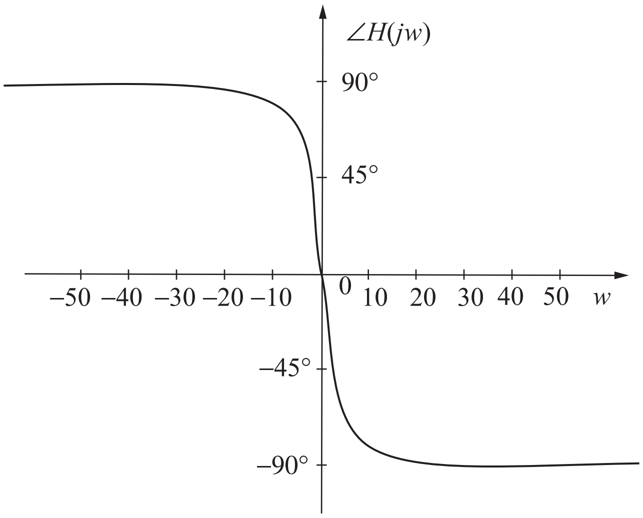


Fig. 2.31. Phase plot

Example 28: For the circuit shown in figure, determine the output voltage $V_0(t)$ to a voltage source excitation $V_{i(t)} = e^{-t}u(t)$ using Fourier transform

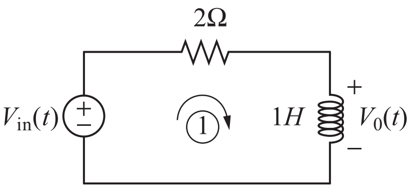


Fig. 2.32.

SOLUTION:

$$\text{Since } V_{in(t)} = e^{-t}u(t) \tag{1}$$

$$V_{in(jw)} = \frac{1}{1+jw} \tag{2}$$

Applying KVL in loop (1)

$$V_{in(t)} = 2i(t) + 1 \cdot \frac{di(t)}{dt}$$

$$V_{in(t)} = 2i(t) + \frac{di(t)}{dt} \tag{3}$$

$$V_0(t) = 1 \cdot \frac{di(t)}{dt}$$

$$V_0(t) = \frac{di(t)}{dt} \tag{4}$$

Q3: (i) State and prove the following properties of Fourier series:
(a) Time shifting property (b) Frequency shifting property
(ii) What are Dirichlet's conditions?

Q4: Find the fundamental period T, the fundamental frequency w_0 and the Fourier series coefficients a_n of the following periodic signal;

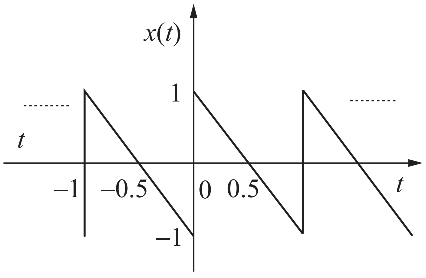


Fig. 2.3 P.

Q5: Obtain the Fourier series component of the periodic square wave signals.

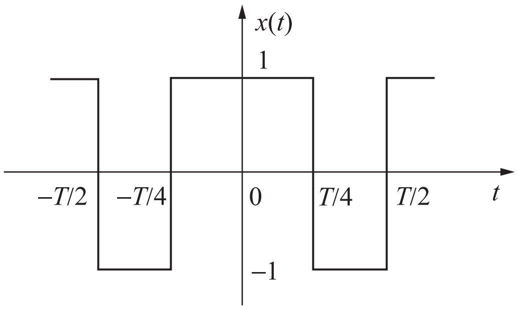


Fig. 2.4 P.

Q6: Determine the Fourier transform of the Gate function

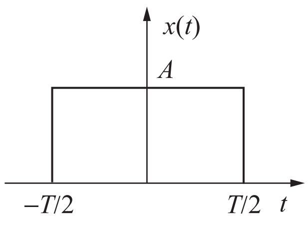


Fig. 2.5 P.

Q7: Determine the Fourier series representation of the signal

$$x(t) = \begin{cases} t - t^2 \text{ for } -\pi \leq t \leq \pi \\ 0 \quad \text{elsewhere} \end{cases}$$

Q8: For the continuous-time periodic signal

$$x(t) = 2 + \cos[2\pi t/3] + 4\sin[5\pi t/3]$$

determine the fundamental frequency w_0 and the Fourier series coefficients C_n such that

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jnw_0 t}$$

Q9: Find the Fourier transform of the following signals:

(a) $x(t) = \delta(t)$ (b) $x(t) = 1$ (c) $x(t) = \text{sgn}\,(t)$ (d) $x(t) = u(t)$

(e) $x(t) = \exp(-at)u(t)$ (f) $x(t) = \cos\,[w_0 t]\sin\,[w_0 t]$

Q10: Show that the Fourier transform of rect $(t-5)$ is $Sa(w/2)\exp(j5w)$. Sketch the resulting amplitude and phase spectrum.

Q11: Find the inverse Fourier transform of spectrum shown in figure.

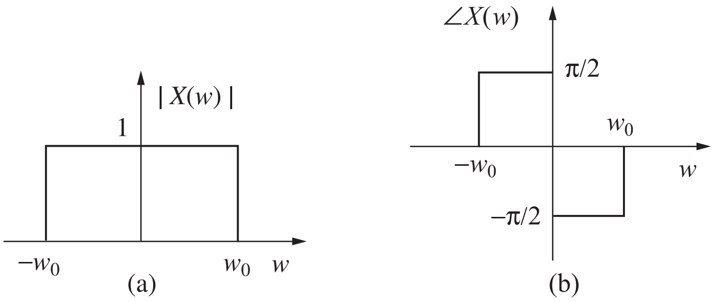


Fig. 2.6 P.

Q12: Find the Fourier transform of the following waveform.

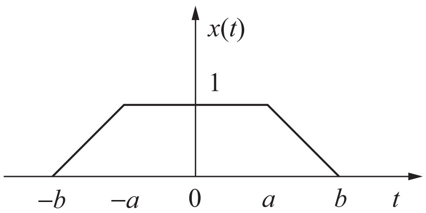


Fig. 2.7 P.

Q13: State and prove duality property of CTFT.

Q14: Determine the Fourier transform of the signal
$x(t) = \{tu(t)^{*}[u(t) - u(t-1)]\}$, where $u(t)$ is unit step function and * denotes the convolution operation.

Q15: Show that the frequency response of a CTLTIS is $Y(w) = H(w)X(w)$

where $X(w) =$ Fourier transform of the signal $x(t)$

$H(w) =$ Fourier transform of LTIS response $h(t)$

Q16: Find the Fourier transform of the signal $x(t)$ shown in figure below.

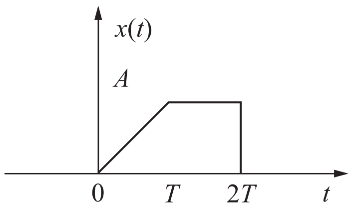


Fig. 2.8 P.

Q17: Determine the frequency response $H(jw)$ and impulse response $h(t)$ for a stable CTLTIS characterized by the linear constant coefficient differential equation given as

$$d^2y(t)/dt^2 + 4dy(t)/dt + 3y(t) = dx(t)/dt + 2x(t)$$

Q18: Find the Fourier transform of the signal $x(t)$ shown in figure below.

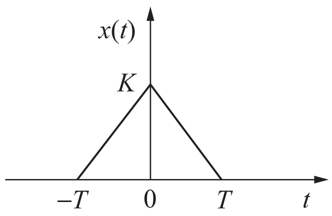


Fig. 2.9 P.

Q19: If $g(t)$ is a complex signal given by $g(t) = g_r(t) + jg_i(t)$ where $g_r(t)$ and $g_i(t)$ are the real and imaginary parts of $g(t)$ respectively. If $G(f)$ is the Fourier transform of $g(t)$, express the Fourier transform of $g_r(t)$ and $g_i(t)$ in terms of $G(f)$.

Q20: Find the coefficients of the complex exponential Fourier series for a half wave rectified sine wave defined by

$$x(t) = \begin{cases} A\sin(w_0 t), \ 0 \le t \le T_0/2 \\ 0, \ T_0/2 \le t \le T_0 \end{cases}$$

$$\text{with } x(t) = x(t + T_0)$$

Q21: (a) Show that the Fourier transform of the convolution of two signals in the time domain can be given by the product of the Fourier transform of the individual signals in the frequency domain.

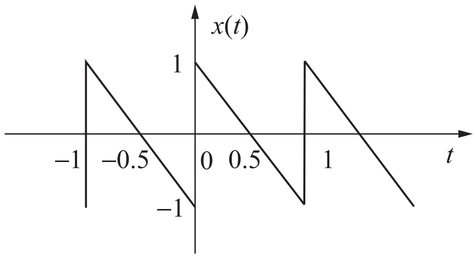
(b) Determine the Fourier transform of the signal

$$x(t) = \frac{1}{2}\left[\delta(t+1) + \delta(t-1) + \delta\left(t + \frac{1}{2}\right)\delta + \left(t - \frac{1}{2}\right)\right]$$

$$a_n = \frac{1-(-1)^n}{n^2\pi^2}$$

$$b_n = \frac{1}{n\pi}$$

Q3:



$$T = 1$$

$$w_0 = 2\pi \text{ rad/sec}$$

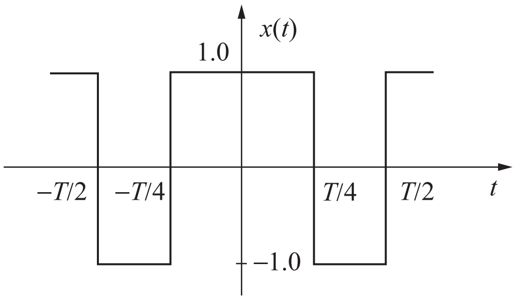
$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$x(t) = -2t + 1$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos n w_0 t\, dt$$

$$a_n = 0$$

Q4:



$$\frac{T}{2} - \left(-\frac{T}{4}\right) = \frac{3T}{4};\ w_0 = \frac{2\pi}{\frac{3T}{4}} = \frac{8\pi}{3T}$$

$$x(t) = \begin{cases} 1\left(-\frac{T}{4} \le t \le \frac{T}{4}\right) \\ -1\left(\frac{T}{4} \le t \le \frac{T}{2}\right) \end{cases}$$

$$a_0 = \frac{1}{\frac{3T}{4}}\left\{\int_{-\frac{T}{4}}^{\frac{T}{4}} dt + \int_{\frac{T}{4}}^{\frac{T}{2}} (-1)dt\right\} = \frac{4}{3T}\frac{T}{4} = \frac{1}{3}$$

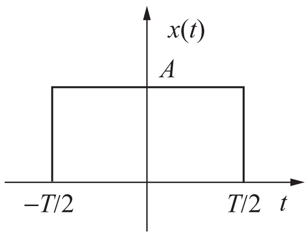
$$a_n = \frac{8}{3T}\left\{\int_{-\frac{T}{4}}^{\frac{T}{4}} \cos\frac{8n\pi}{3T}dt - \int_{\frac{T}{4}}^{\frac{T}{2}} \cos\frac{8n\pi}{3T}t\,dt\right\}$$

$$a_n = \frac{1}{n\pi}\left[3\sin\frac{2n\pi}{3} - \sin\frac{4n\pi}{3}\right]$$

$$b_n = 0,\ \text{since even function}$$

$$x(t) = \frac{1}{3} + \frac{1}{\pi}\left[3\sin\frac{2\pi}{3} - \sin\frac{4\pi}{3} + \frac{3}{2}\sin\frac{4\pi}{3} - \frac{1}{2}\sin\frac{8\pi}{3} + \cdots\right]$$

Q5:



$$x(t) = \begin{cases} \text{A} - \frac{T}{2} \le t \le \frac{T}{2} \\ 0 \quad \text{elsewhere} \end{cases}$$

$$X(jw) = A\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-jwt}dt = \frac{2A}{w}\sin\frac{wT}{2} = \frac{AT}{\frac{wT}{2}}\sin\frac{wT}{2}$$

$$X(if) = AT\,\text{sin}c\,fT$$

Q6:

$$T_0 = 2\pi;$$

$$w_0 = 1;$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(t - t^2\right)dt = \frac{-\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(t - t^2\right)\cos nt\,dt = \frac{-4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(t - t^2\right)\sin nt\,dt = \frac{-2(-1)^n}{n}$$

Taking inverse Fourier transform

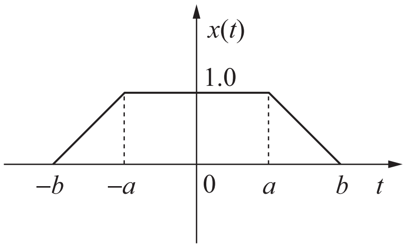
$$x_1(t) = \frac{1}{2\pi}\int_0^{w_0} -j\, e^{jwt} dw = \frac{1 - e^{jw_0 t}}{2\pi t}$$

$$x_2(t) = \frac{1}{2\pi}\int_{-w_0}^{0} j\, e^{jwt} dw = \frac{1 - e^{-jw_0 t}}{2\pi t}$$

$$x(t) = x_1(t) + x_2(t) = \frac{1}{2\pi t}(1 - e^{jw_0 t} + 1 - e^{-jw_0 t})$$

$$= \frac{1}{2\pi t}(2 - 2\cos w_0 t) = \frac{2\sin^2 \frac{w_0 t}{2}}{\pi t}$$

Q11:



$$x(t) = \begin{cases} \frac{t+b}{b-a} & \text{for} -b < t < -a \\ 1 & \text{for} -a < t < a \\ \frac{t-b}{a-b} & \text{for} \quad a < t < b \end{cases}$$

$$X(jw) = \frac{2}{w^2(b-a)}(\cos wa - \cos wb)$$

Q12:

$$x(t) = tu(t)^*[u(t) - u(t-1)]$$

$$x_1(t) = tu(t) \qquad x_2(t) = u(t) - u(t-1)$$

Differentiating in frequency domain property

$$\text{FT}(tx(t)) = j\frac{d}{dw}X(jw)$$

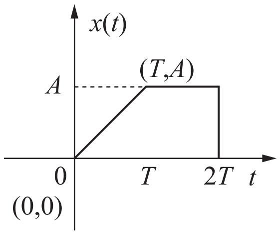
$$X_1(jw) = \frac{1}{(jw)^2}$$

$$X_2(jw) = \int_0^1 1.e^{-jwt} dt = \frac{1}{jw}(1 - e^{-jw})$$

$$X(jw) = X_1(jw)X_2(jw) = \frac{1}{(jw)^3}(1 - e^{-jw})$$

Q13: Prove convolution in time domain property.

Q14:



$$x(t) = \begin{cases} \frac{A}{T}t & 0 < t < T \\ A & T < t < 2T \end{cases}$$

$$X(jw) = \frac{A}{T}\int_0^T te^{-jwt}dt + A\int_T^{2T} e^{-jwt}dt$$

$$X(jw) = \frac{A}{T}\left[\frac{te^{-jwt}}{-jw}\int_0^T - \int_0^T \frac{e^{-jwt}}{jw}dt\right] + A\left[\frac{e^{-jwt}}{-jw}\bigg| \begin{matrix} 2T \\ T \end{matrix}\right]$$

$$= \frac{A}{T}\left\{\frac{Te^{jwt}}{-jw} + \frac{1}{w^2}\left(e^{-jwT} - 1\right)\right\} + A\left\{\frac{e^{-j2wT} - e^{-jwT}}{-jw}\right\}$$

$$= A\left\{\frac{e^{-jwt}}{-jw} + \frac{1}{w^2T}\left(e^{-jwT} - 1\right)\right\} - \frac{A}{jw}e^{-jwT}\left(e^{-jwT} - 1\right)$$

$$= \frac{Ae^{-jw}T}{jw} + \frac{A}{w^2T}e^{-jwT} - \frac{A}{w^2T} - \frac{A}{jw}e^{-j2wT} + \frac{A}{jw}e^{-jwT}$$

$$= \frac{A}{wT}\left(\frac{1}{w}e^{-jwT} - \frac{1}{w} + jTe^{-2jwT}\right)$$

Q15:

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t) \quad (1)$$

Taking Fourier transform on both sides

$$(jw)^2Y(jw) + 4(jw)Y(jw) + 3Y(jw) = (jw)X(jw) + 2X(jw)$$

$$\left((jw)^2 + 4(jw) + 3\right)Y(jw) = ((jw) + 2)X(jw) \quad (2)$$

Frequency response $H(jw) = \frac{Y(jw)}{X(jw)} = \frac{2 + jw}{(jw)^2 + 4jw + 3}$ (3)

$$H(jw) = \frac{2 + jw}{(3 + jw)(1 + jw)} = \frac{A}{3 + jw} + \frac{B}{1 + jw}$$

$$x(t) = A\sin w_0 t \text{ for } 0 \le t \le \frac{T_0}{2}$$

$$= 0 \qquad \text{for } \frac{T_0}{2} \le t \le T_0$$

$$C_0 = \frac{1}{T_0}\int_0^{\frac{T_0}{2}} A\sin w_0 t\,dt = \frac{A}{T_0}\left(\frac{-\cos w_0 t}{w_0}\,\Bigg|\,_0^{\frac{T_0}{2}}\right)$$

$$= -\frac{A}{T_0 \cdot \frac{2\pi}{T_0}}\left[\cos w_0 \cdot \frac{T_0}{2} - 1\right] = -\frac{A}{2\pi}[\cos\pi - 1] = \frac{A}{2}$$

$$C_n = \frac{1}{T_0}\int_0^{\frac{T_0}{2}} A\sin w_0 t e^{-jnw_0 t}\,dt$$

$$= \frac{A}{2jT_0}\int_0^{\frac{T_0}{2}} (e^{jw_0 t} - e^{-jnw_0 t})e^{-jnw_0 t}\,dt$$

$$= \frac{A}{2jT_0}\int_0^{\frac{T_0}{2}} \left(e^{jw_0 t(1-n)} - e^{-jw_0 t(n+1)}\right)dt$$

$$= \frac{A}{2jT_0}\left(\frac{e^{jw_0 t(1-n)}}{jw_0(1-n)} - \frac{e^{-jw_0 t(n+1)}}{-jw_0(n+1)}\,\Bigg|\,_0^{\frac{T_0}{2}}\right)$$

$$= \frac{A}{2jT_0 w_0}\left(\frac{e^{jw_0(1-n)\frac{T_0}{2}}}{1-n} + \frac{e^{-jw_0(n+1)\frac{T_0}{2}}}{(n+1)} - \frac{1}{1-n} - \frac{1}{n+1}\right)$$

$$= -\frac{A}{4\pi}\left[\frac{e^{j\pi(1-n)}}{1-n} + \frac{e^{-j\pi(n+1)}}{n+1} - \frac{1}{1-n} - \frac{1}{n+1}\right]$$

$$= -\frac{A}{4\pi}\left(\frac{e^{j\pi}e^{-jn\pi}}{1-n} + \frac{e^{-jn\pi} \cdot e^{-j\pi}}{n+1} - \frac{1}{1-n} - \frac{1}{n+1}\right)$$

$$= -\frac{A}{4\pi}\left(\frac{-e^{-jn\pi}}{1-n} - \frac{e^{-jn\pi}}{n+1} - \frac{1}{1-n} - \frac{1}{n+1}\right) \qquad \text{Since } e^{j\pi} = -1$$

$$= \frac{A}{4\pi}\left(\frac{2e^{-jn\pi}}{1-n^2} + \frac{2}{1-n^2}\right)$$

$$= \frac{A}{2\pi(1-n^2)}(e^{-jn\pi} + 1)$$

Q19:

$$x(t) = \frac{1}{2}\left(\delta(t+1) + \delta(t-1) + \delta\left(t + \frac{1}{2}\right) + \delta\left(t - \frac{1}{2}\right)\right)$$

Taking Fourier transform on both sides

$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt \qquad (1)$$

$$X(jw) = \int_{-\infty}^{\infty} \frac{1}{2}\left(\delta(t+1) + \delta(t-1) + \delta\left(t + \frac{1}{2}\right) + \delta\left(t - \frac{1}{2}\right)\right)e^{-jwt}dt$$

$$X(jw) = \frac{1}{2}\left(\int_{-\infty}^{\infty} \delta(t+1)e^{-jwt}dt + \int_{-\infty}^{\infty} \delta(t-1)e^{-jwt}dt + \int_{-\infty}^{\infty} \delta\left(t + \frac{1}{2}\right)e^{-jwt}dt + \int_{-\infty}^{\infty} \delta\left(t - \frac{1}{2}\right)e^{-jwt}dt\right)$$

Since $\mathrm{FT}(\delta(t)) = 1$

So $\mathrm{FT}(\delta(t \pm t_0)) = e^{\pm jwt_0}dt$ {using time shifting property}

$$X(jw) = \frac{1}{2}\left(e^{jw} + e^{-jw} + e^{j\frac{w}{2}} + e^{-j\frac{w}{2}}\right)$$

$$X(jw) = \frac{e^{jw} + e^{-jw}}{2} + \frac{e^{j\frac{w}{2}} + e^{-j\frac{w}{2}}}{2}$$

$$X(jw) = \cos\, w + \cos\frac{w}{2}$$

OBJECTIVE TYPE QUESTIONS

Q1: If the Fourier transform of a function $x(t)$ is $X(jw)$, then $X(jw)$ is defined as

(a) $\int_{-\infty}^{\infty} x(t)e^{jwt}dt$ (b) $\int_{-\infty}^{\infty} \frac{dx(t)}{dt}e^{-jwt}dt$
(c) $\int_{-\infty}^{\infty} x(t)dt$ (d) $\int_{-\infty}^{\infty} x(t)e^{-jwt}dt$

Q2: If $X(jw)$ be the Fourier transform of $x(t)$, then

(a) $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{jwt}dw$ (b) $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{-jwt}dw$
(c) $x(t) = \frac{1}{2\pi}\int_{0}^{\infty} X(jw)e^{jwt}dw$ (d) $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{-jwt}dw$

Q3: Fourier transform of $x(t) = 1$ is

(a) $2\pi\,\delta(w)$ (b) $\pi\,\delta(w)$ (c) $3\pi\,\delta(w)$ (d) $4\pi\,\delta(w)$

Q4: Fourier transform of $x(t - t_0)$ is

(a) $e^{-jwt}{}_0X(jw)$ (b) $e^{jwt}{}_0X(jw)$ (c) $\frac{1}{t_0}X(jw)$ (d) $t_0e^{-jwt}{}_0X(jw)$

Q19: The trigonometric Fourier series of a periodic time function have

(a) sine terms (b) cosine term

(c) both (a) and (b) (d) DC term

Q20: Fourier series is defined as

$x(t) = a_\circ + \sum_{n=1}^{\infty} (a_n \cos nw_0t + b_n \sin w_0t)$

(a) True (b) False

Answers:

(1) d	(2) a	(3) a	(4) a	(5) b
(6) c	(7) a	(8) a	(9) a	(10) d
(11) c	(12) b	(13) b	(14) a	(15) a
(16) a	(17) e	(18) c	(19) c	(20) a

UNSOLVED PROBLEMS

Q1: Show that the Fourier transform of $x(t) = \delta(t+2) + \delta(t) + \delta(t-2)$ is $(1 + 2\cos 2w)$.

Q2: Show that the inverse Fourier transform of $X(jw) = 2\pi\delta(w) + \pi\delta(w-4\pi) + \pi\delta(w+4\pi)$ is $x(t) = 1 + \cos\ 4\pi t$.

Q3: Calculate the Fourier transform of $te^{-|t|}$, using the F.T. pair, FT $\left[e^{-|t|}\right] = \frac{2}{1+w^2}$. Also find the Fourier transform of $\frac{4t}{(1+t^2)^2}$ using duality property.

Q4: $X(jw) = \delta(w) + \delta(w-\pi) + \delta(w-5)$; find IFT $x(t)$ and show that $x(t)$ is non-periodic.

Q5: Find the Fourier transform of the triangular pulse as shown in figure.

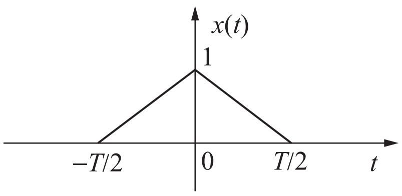


Fig. 2.10 P.

Ans. $X(jw) = \frac{T}{2} \sin c^2(\frac{wt}{4})$

Q6: Find the Fourier transform of $x(t) = \text{rect}(t/2)$. Ans. $X(jw) = 2\sin cw$

Q7: Find the Fourier transform of the signal $x(t) = \cos w_0t$ by using the frequency shifting property.

Ans: $X(jw) = \pi[\delta(w-w_0) + \delta(w+w_0)]$

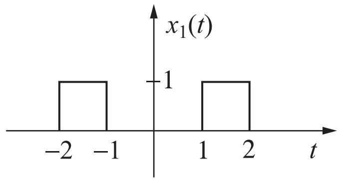
Q8: Show that FT $[\sin w_0tu(t)] = \frac{w_0}{w_0^2 - w^2} + \frac{\pi j}{2}[\delta(w+w_0) - \delta(w-w_0)]$.

Q9: Find inverse Fourier transform of $X(jw) = \frac{jw}{(1+jw)^2}$

Ans. $x(t) = \frac{d}{dt}[te^{-t}u(t)]$

Q10: Sketch and then find the Fourier transform of following signals

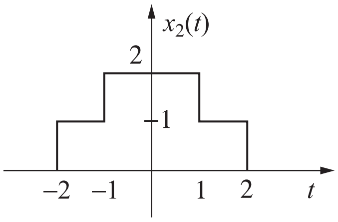
(a) $x_1(t) = \pi\left(t+\frac{3}{2}\right) + \pi\left(t-\frac{3}{2}\right)$ Ans. (a)



$X_1(jw) = 2\sin c\frac{w}{2}\cos 3\frac{w}{2}$

Fig. 2.11 P.

(b) $x_2(t) = \pi\left(\frac{t}{4}\right) + \pi\left(\frac{t}{2}\right)$ Ans. (b)



$X_2(jw) = 4\sin c2w + 2\sin cw$

Fig. 2.12 P.

Q11: Find the frequency response $x(jw)$ of the RC circuit shown in figure. Plot the magnitude and phase response for $RC = 1$

$$x(jw) = \frac{y(jw)}{x(jw)} = \frac{1}{1+jwRC}$$

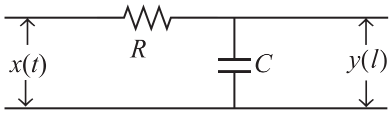


Fig. 2.13 P.

Ans. $|x(jw)| = \dfrac{1}{\sqrt{1+w^2}}$

$x(jw) = -\tan^{-1} w$

Q12: Find the Fourier series of the waveform shown in figure.

$$x(t) = \frac{2A}{jn\pi} \text{for } n = 1,3,5,7$$

$$x(t\pi)t = \frac{1}{\pi} + \frac{1}{2}\cos 5\pi t - \frac{2}{3\pi}\cos 10$$
$$-\frac{2}{8\pi}\cos 15\pi t$$

Q15: The output of a system is given by

$$x(t) = \begin{cases} A\sin w_0 t & \text{for} \quad 0 \leq t \leq \pi \\ 0 & \text{for} \quad \pi \leq t \leq 2\pi \end{cases}$$

Determine trigonometric form of Fourier series of $x(t)$

Ans. $\left[x(t) = \frac{A}{\pi} + \frac{A}{2}\cos(nt - \frac{\pi}{2}) + \sum_{n=2}^{\infty} \frac{2A}{\pi(1-n^2)}\cos nt\right]$